Prescribing the length of a de Rham curve

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Abstract

The length L of the de Rham curve is the common limit of two monotonic sequences of lengths (l^n) and (L^n) of inscribed and circumscribed polygons respectively depending on a parameter γ.
In this paper we produce a family of de Rham curves subject to certain constraints as interpolation and convexity. The arc length function depends on two parameters and is convex. We propose an algorithm to get a prescribed length.

1. Introduction

The de Rham curve C_γ, studied in [3], is the limit of a sequence of polygons depending on a parameter γ.
In section 2, we recall the construction of the curve C_γ and its first properties. We are interested in the computation of the length L of this curve. First we define upper and lower approximations of L as the lengths of two approximating polygons. They form two sequences (L^n) and (l^n) which both converge monotonically to L. We recall the convergence speed of (L^n) to L for all $\gamma > 1$ which suggests the possibility of accelerating the convergence of the two sequences (L^n) and (l^n) by the ε-algorithm or the iterated Aitken's Δ^2 algorithm (see e.g. chapter 2 of [1]).
In section 3, we build a family of convex interpolating curves with variable length $L(x, y)$. We study the properties of the function L: convexity and monotonicity of $L(x, x)$. In section 4, we propose an algorithm to reach a prescribed length. Section 5 is devoted to numerical results. This problem was already considered by other authors in the context of computer-aided geometric design, in particular for piecewise polynomial or rational curves

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(see e.g. [4],[5]).

2. Construction and properties of the de Rham curve

Let ABC be a triangle. The curve is the limit of a sequence of polygons, $(P^n), n = 0, 1, 2, \ldots$ starting with $P^0 = \{A, B, C\}$. Then the points dividing into three parts the sides of the polygon P^n obtained at the n-th step are the vertices of the next one. The three parts have lengths proportional to $1, \gamma, 1$ respectively. The number of sides of P^n is $2^n + 1$.

We denote by $S_0^n, S_1^n, \ldots, S_{2^n+1}^n$ the vertices of P^n. The construction of de Rham in order to get the next polygon $P^{n+1} = \{S_0^{n+1}, S_1^{n+1}, \ldots, S_{2^{n+1}+1}^{n+1}\}$ from the previous one P^n is as follows: $S_{2i}^{n+1} = \alpha S_i^n + \beta S_{i+1}^n$ and $S_{2i+1}^{n+1} = \beta S_i^n + \alpha S_{i+1}^n$ for $i = 0 \ldots 2^n$, where $\beta = 1/(\gamma + 2), \alpha = 1 - \beta$.

In Fig. 1, we show the first step in the construction of de Rham with $P^0 = \{A, B, C\}$ and $P^1 = \{A', B', C', D'\}$.

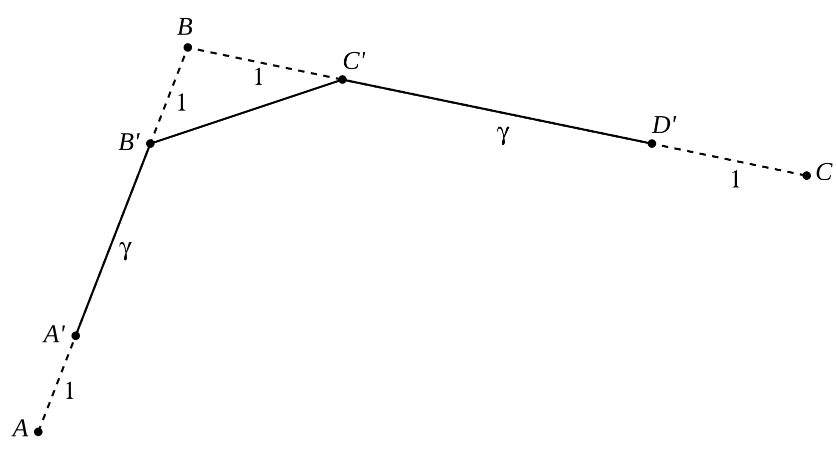

figure 1: The construction of the de Rham curve.

The following properties are given by de Rham in [3].

- The polygons P^n are convex and the sequence (P^n) converges to a curve C_γ which is continuous and convex.
- C_γ is tangent at the midpoint of each side of P^n.
- If $\gamma > 1$, C_γ has a tangent at each point and the slope is continuous.
- For $\gamma = 2$, C_2 is an arc of a parabola from the midpoint of $[AB]$ to the midpoint of $[BC]$.

From now on, we shall suppose $\gamma > 1$.

We denote by $M_0^n, M_1^n, \ldots, M_{2^n}^n$ the midpoints of the sides of P^n. Let L^n be the length of P^n measured from the midpoint M_0^n of the first side to the midpoint $M_{2^n}^n$ of the last one, and let l^n be the length of the polygonal line joining the midpoints: $M_0^n M_1^n \ldots M_{2^n}^n$.With these notations, $M_i^n = (S_i^n + S_{i+1}^n)/2$ and $M_{2i}^{n+1} = M_i^n$. We write $|U|$ for the euclidean norm of the vector U. Thus, we have
$L^0 = |M_0^0 S_1^0| + |S_1^0 M_1^0| = (|AB| + |BC|)/2$ and $l^0 = |AC|/2$.
$\forall n \in \mathbb{N},\ L^n = \sum_{i=0}^{2^n-1} |M_i^n S_{i+1}^n| + |S_{i+1}^n M_{i+1}^n|$ and $l^n = \displaystyle\sum_{i=0}^{2^n-1} |M_i^n M_{i+1}^n|$.

We have proved the following results in [2]:

- $\forall n \in \mathbb{N},\ L^{n+1} = \dfrac{\gamma L^n + 2l^n}{\gamma + 2}$
- The two sequences (L^n) and (l^n) are respectively decreasing and increasing and they converge to the same limit L, which is the length of C_γ.
- There exists $c \in \mathbb{R}_+^*$ and $\kappa \in]0,1[$ such that $\forall n \in \mathbb{N}$, $|L^{n+1} - L^n| \leq c.\kappa^n$.
- For $\gamma = 2$, let $S_0 = A, S_1 = B, S_3 = C$, $\Delta S_i = S_{i+1} - S_i$ and $\Delta^2 S_i = \Delta S_{i+1} - \Delta S_i$. In this particular case, we are able to evaluate exactly the length L of the curve and to estimate precisely the convergence rate of the sequences (L^n).

$$L = \frac{|\Delta S_0|^2 |\Delta S_1|^2 - (\Delta S_0.\Delta S_1)^2}{|\Delta^2 S_0|^3} \ln\left(\frac{\Delta S_1.\Delta^2 S_0 + |\Delta S_1||\Delta^2 S_0|}{\Delta S_0.\Delta^2 S_0 + |\Delta S_0||\Delta^2 S_0|}\right)$$

$$+ \quad \frac{|\Delta S_1|(\Delta S_1.\Delta^2 S_0) - |\Delta S_0|(\Delta S_0.\Delta^2 S_0)}{|\Delta^2 S_0|^2}.$$

$$\text{and} \lim_{n\to+\infty} \frac{L^{n+1} - L}{L^n - L} = \lim_{n\to+\infty} \frac{L^{n+1} - L^n}{L^n - L^{n-1}} = \frac{1}{4}.$$

3. Interpolation and length

The curve C_γ may be seen as an interpolating convex curve from the midpoint of the initial side $[A, B]$ with tangent in the direction of the vector AB to the midpoint of $[B, C]$ with tangent in the direction of BC.

Conversely, let us consider two points A and C in the plane and two corresponding vectors u_A and u_C such that the lines δ_A and δ_C through A and C and parallel to u_A and u_C respectively intersect in a unique point B of the form $B = A + \alpha u_A = C - \beta u_C$ with $\alpha > 0$ and $\beta > 0$ (see fig2). Given a length L such that $|AC| < L < |AB| + |BC|$, we wish to built a convex curve of length L, interpolating the data (A, u_A) and (C, u_C).

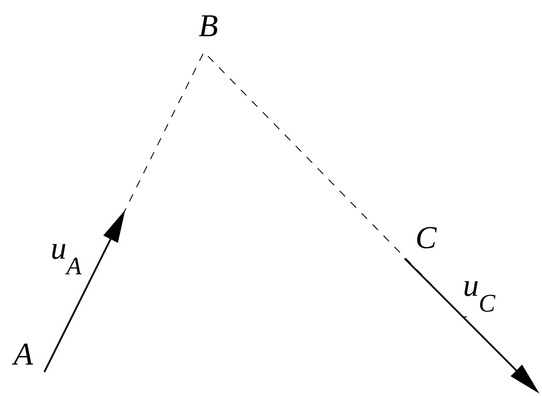

figure 2

This could probably be done by choosing a good γ for the construction of the curve but we have another choice. We set γ, with $\gamma > 1$.
For $(x, y) \in [0, 1]^2$, let us define $S_0(x), S_1(x), S_2(y), S_3(y)$ by:

$$S_1(x) = xB + (1 - x)A, S_0(x) = 2A - S_1(x),$$
$$S_2(y) = yB + (1 - y)C, S_3(y) = 2C - S_2(y).$$

so that $S_1(x)$ is inserted in $[A, B]$ and $S_2(y)$ in $[B, C]$, A is the midpoint of $[S_0(x), S_1(x)]$ and C the midpoint of $[S_2(y), S_3(y)]$.
Then, let us built the two de Rham curves on the triangles $\{S_0(x), S_1(x), S_2(y)\}$ and $\{S_1x), S_2(y), S_3(y)\}$. The first one joins A to $I(x, y)$, the midpoint of $[S_1(x), S_2(y)]$, the second one joins $I(x, y)$ to C. Moreover, the tangent at A is in the direction u_A and similarly in C. In $I(x, y)$ both curves have a tangent in the direction of the vector $S_1(x)S_2(y)$. As the initial polygon is convex, the union of the two curves gives a convex curve interpolating the data for all $(x, y) \in]0, 1[^2$. Its length is $L(x, y) = L_1(x, y) + L_2(x, y)$, where L_1 (respectively L_2) is the length of the first curve (resp. the second curve). Note that these triangles are degenerate if x or y is 0 or 1, but we can still construct the curve. We shall now study the properties of the function L which is depending on γ.

Proposition 1 *For any γ, L is a convex function on $[0, 1]^2$ and so L is continuous.*

Proof : We shall prove that L_1 is convex. Similarly L_2 will be convex, so we shall have the property. The length of the polygonal line at step n in the construction of the first curve will be noted $L_1^n(x, y)$.
Let $\{a, b, c\}$ be a referential triangle. We define the affine transformation $\varphi(x, y)$ by:

$$\varphi(x, y)(a) = S_0(x), \varphi(x, y)(b) = S_1(x), \varphi(x, y)(c) = S_2(y).$$

In the construction of the de Rham curve, from step n to step $n + 1$, we have $S_{2i}^{n+1} = \alpha S_i^n + \beta S_{i+1}^n$ and $S_{2i+1}^{n+1} = \beta S_i^n + \alpha S_{i+1}^n$ for $i = 0 \dots 2^n$, where $\beta = 1/(\gamma + 2), \alpha = 1 - \beta$. So that if we denote σ_i^n a vertex of the polygon π^n, built in the referential triangle, we get:

$$\forall n \in \mathrm{IN}, \forall \mathrm{i} \in \{0, \dots, 2^{\mathrm{n}} + 1\}, \varphi(\mathrm{x}, \mathrm{y})(\sigma_{\mathrm{i}}^{\mathrm{n}}) = \mathrm{S}_{\mathrm{i}}^{\mathrm{n}}.$$

Now let m and m' be two points of the plane. Using the barycentric coordinates
$m = ua + vb + wc$ with $u + v + w = 1$ and $m' = u'a + v'b + w'c$ with $u' + v' + w' = 1$, we get:

$$\begin{aligned} |\varphi(x, y)(m)\varphi(x, y)(m')| &= |(u - u')\varphi(x, y)(a) + (v - v')\varphi(x, y)(b) + (w - w')\varphi(c)| \\ &= |(u - u')S_0(x) + (v - v')S_1(x) + (w - w')S_2(y)|. \end{aligned}$$

Let λ_1 and λ_2 be two real numbers in $[0,1]$ with $\lambda_1+\lambda_2=1$, and let (x_1,y_1) and (x_2,y_2) be two couples in $[0,1]^2$.
By construction of S_0, we have $S_0(\lambda_1 x_1+\lambda_2 x_2)=\lambda_1 S_0(x_1)+\lambda_2 S_0(x_2)$ and a similar property holds for S_1 and S_2, so that

$$\begin{aligned}
& |\varphi(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))(m)\varphi(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))(m')| \\
= & |(u-u')S_0(\lambda_1 x_1+\lambda_2 x_2)+(v-v')S_1(\lambda_1 x_1+\lambda_2 x_2)+(w-w')S_2(\lambda_1 y_1+\lambda_2 y_2)| \\
= & |\lambda_1((u-u')S_0(x_1)+(v-v')S_1(x_1)+(w-w')S_2(y_1)) \\
+ & \lambda_2((u-u')S_0(x_2)+(v-v')S_1(x_2)+(w-w')S_2(y_2))| \\
\leq & |\lambda_1((u-u')S_0(x_1)+(v-v')S_1(x_1)+(w-w')S_2(y_1))| \\
+ & |\lambda_2((u-u')S_0(x_2)+(v-v')S_1(x_2)+(w-w')S_2(y_2))| \\
= & \lambda_1|\varphi(x_1,y_1)(m)\varphi(x_1,y_1)(m')|+\lambda_2|\varphi(x_2,y_2)(m)\varphi(x_2,y_2)(m')|.
\end{aligned}$$

Now, remember the definition of L^n in the previous section.

$$\begin{aligned}
L_1^n(x,y) &= |S_0^n S_1^n|/2+\sum_{i=1}^{2^n-1}|S_i^n S_{i+1}^n|+|S_{2^n}^n S_{2^n+1}^n|/2 \\
&= |\varphi(x,y)(\sigma_0^n)\varphi(x,y)(\sigma_1^n)|/2 \\
&+ \sum_{i=1}^{2^n-1}|\varphi(x,y)(\sigma_i^n)\varphi(x,y)(\sigma_{i+1}^n)| \\
&+ |\varphi(x,y)(\sigma_{2^n}^n)\varphi(x,y)(\sigma_{2^n+1}^n)|/2.
\end{aligned}$$

Using the above result, we easily obtain:

$$L_1^n(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))\leq\lambda_1 L_1^n(x_1,y_1)+\lambda_2 L_1^n(x_2,y_2).$$

This allows us to conclude the proof as n tends to $+\infty$. ■

Proposition 2 *L takes all values between $|AC|$ and $|AB|+|BC|$.*

Proof : $L(0,0)=|AC|$ and $L(1,1)=|AB|+|BC|$. As L is a continuous function, we get the result. This result is independent of γ. ■

Proposition 3 *The function f, defined by $f(x)=L(x,x)$ is a strictly increasing convex function on $[0,1]$.*

Proof : As L is convex, f is convex.
Now $f(0) = |AC|$ and if $x > 0, L(x,x) > l^0(x,x) = |AC|$. We already know that (l^n) is a strictly increasing sequence converging to L. As f is convex, the function ψ defined by $\psi(t) = \dfrac{f(x) - f(t)}{x - t}$ is increasing on $[0,1] - \{x\}$. If $x > x' > 0$ then $\psi(x') \geq \psi(0) > 0$ so that $f(x) > f(x')$. ■

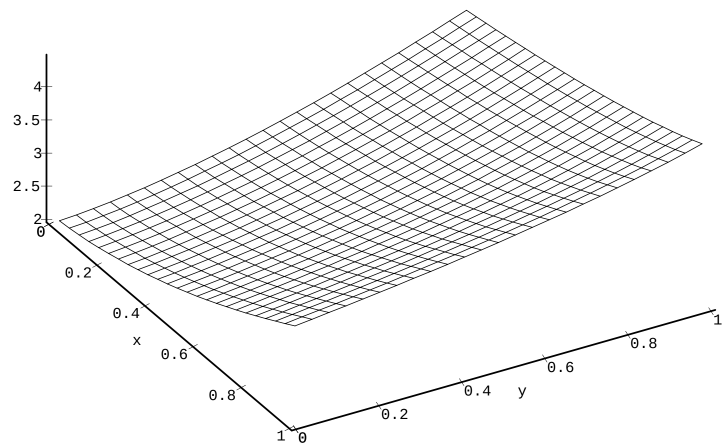

figure 3: The function L for $\gamma = 2$.

4. Algorithms giving a curve with a prescribed length

Let A, B, C be a triangle and choose $\gamma > 1$. We want to get a de Rham interpolating curve of length $\bar{l}$, $|AC| < \bar{l} < |AB| + |BC|$. Precisely, using the function f defined above, we have to solve the equation $f(x) = L(x,x) = \bar{l}$. We need lemmas about the regula falsi and secant methods. We only prove the second one, as the first one is easy to prove.

Lemma 4 *If g is a convex function on $[0,1]$ with $g(0) < 0 < g(1)$, then the sequence (x_p) where $x_0 = 0$, $x_{p+1} = 1 - g(1)\dfrac{x_p - 1}{g(x_p) - g(1)}$ for $p \geq 0$, built by the regula falsi method converges to the unique solution of $g(x) = 0$.*

Lemma 5 *Let g be a strictly increasing convex function on $[0,1]$ with $g(0) < 0 < g(1)$. We build a sequence (x_p) by a modified secant method:*
$x_0 = 0$, $x_1 = 1$,
and if $u = x_p - g(x_p)\dfrac{x_p - x_{p-1}}{g(x_p) - g(x_{p-1})}$, then

$$x_{p+1} = \begin{cases} u \text{ for } p = 3k-2 \text{ or } p = 3k-1 \\ \min(u, x_{3k'+1}, k' = 0 \ldots k-1) \text{ for } p = 3k. \end{cases}$$

Then the sequence $x_0, x_2, x_3, \ldots, x_{3k-1}, x_{3k}, \ldots$ converges to the unique solution of $g(x) = 0$.

Proof : As $g(0) < 0 < g(1)$ and g is continuous and stricly increasing, the equation $g(x) = 0$ has a unique solution $\overline{x}$.
First, we prove that if $x_{p-1} < \overline{x} < x_p$ then $x_{p-1} < u \leq \overline{x}$.
Indeed, with the hypothesis $g(x_{p-1}) < 0 < g(x_p)$.
Now $u = x_p - g(x_p)\dfrac{x_p - x_{p-1}}{g(x_p) - g(x_{p-1})} = x_{p-1} - g(x_{p-1})\dfrac{x_p - x_{p-1}}{g(x_p) - g(x_{p-1})}$;
as $g(x_{p-1}) < 0$ and $\dfrac{x_p - x_{p-1}}{g(x_p) - g(x_{p-1})} > 0$, we get $u > x_{p-1}$.
Then $u = \dfrac{g(x_p)}{g(x_p) - g(x_{p-1})}x_{p-1} + (1 - \dfrac{g(x_p)}{g(x_p) - g(x_{p-1})})x_p$,
Using the convexity of g, as $0 < \dfrac{g(x_p)}{g(x_p) - g(x_{p-1})} < 1$, we have:

$$g(u) \leq \frac{g(x_p)}{g(x_p) - g(x_{p-1})}g(x_{p-1}) + (1 - \frac{g(x_p)}{g(x_p) - g(x_{p-1})})g(x_p) = 0,$$

so that $u \leq \overline{x}$.
Similarly if $x_p < \overline{x} < x_{p-1}$ then $x_p < u < \overline{x}$ and if $x_{p-1} < x_p < \overline{x}$ then $\overline{x} < u$.
As $x_0 < \overline{x} < x_1$, by induction, the two sequences $x_0, x_2, x_3, \ldots, x_{3k-1}, x_{3k}, \ldots$ and (x_{3k+1}) are respectively increasing and decreasing and bounded, so that they converge respectively to x and x' with $x \leq \overline{x} \leq x'$.
If $x < \overline{x}$, by continuity of g, we have $x = x' - g(x')\dfrac{x - x'}{g(x) - g(x')}$ so that

$x' - x = g(x')\dfrac{x - x'}{g(x) - g(x')}$, hence $g(x) = 0$ and $x = \overline{x}$.
We can conclude that the sequence $x_0, x_2, x_3, \ldots, x_{3k-1}, x_{3k}, \ldots$ converges to $\overline{x}$. ■

With these lemmas, we are ready to study the algorithm. Practically, we are not able to evaluate the expression $f(x) = L(x,x)$ except for $\gamma = 2$. Instead, we evaluate $f^n(x) = L^n(x,x)$, the length of the polygonal line obtained at step n. The function f^n is convex and stricly increasing; moreover $f^n(0) = |AC|$ and $f^n(1) = |AB| + |BC|$. We have to choose n to control the approximation. We know that $|L - L^n| \leq c.\kappa^n$. Numerical evaluations have shown that $c \leq |AB| + |BC|$ and $\kappa \approx 0.25$. When n is fixed, we apply the regula falsi or secant method to $g = f^n - \overline{l}$. Note that if we accelerate the evaluation of $f^n(x)$ by an extrapolation algorithm, we cannot evaluate the errors.

algorithm:

- *Given* $\overline{l}$ *and a precision* ε
- *Choose* n *such that* $(|AB| + |BC|) * 0.25^n < \varepsilon/2$.
- *Build the sequence* (x_p^n) *by the regula falsi (or secant) method applied to* $f^n - \overline{l}$ *until* $|f^n(x_p) - \overline{l}| \leq \varepsilon/2$.
- *Then* $|f(x_p) - \overline{l}| \leq |f(x_p) - f^n(x_p)| + |f^n(x_p) - \overline{l}| \leq \varepsilon$.

5. Numerical results

In an orthonormal basis, let $A = (-1.5, -1)$, $B = (0, 1)$ and $C = (1.5, 0)$, so that $|AC| = \sqrt{10} \approx 3.162277766$ and $|AB|+|BC| = \dfrac{5 + \sqrt{13}}{2} \approx 4.302275637$. To verify the algorithms, we have chosen $\gamma = 2$ to compare the approximate value $f^n(x)$ to the exact one, $f(x)$.

ε	n	$\bar{l}$	p	x_p	$f^n(x_p)$	$f(x_p)$
10^{-2}	5	3.2	5	0.0954	3.1956	3.1955
10^{-4}	8	3.2	13	0.103532	3.1999662	3.1999639
$2.2 * 10^{-6}$	10	3.2	19	0.1035945	3.19999915	3.19999901
10^{-2}	5	4	3	0.8242	3.9987	3.9985
10^{-4}	8	4	5	0.825195	3.9999861	3.9999833
$2.2 * 10^{-6}$	10	4	7	0.8252054	3.99999986	3.99999968
10^{-2}	5	4.25	2	0.9722	4.2490	4.2488
10^{-4}	8	4.25	3	0.972791	4.2499653	4.2499624
$2.2 * 10^{-6}$	10	4.25	5	0.9728105	4.24999993	4.24999978

regula falsi method

ε	n	$\bar{l}$	p	x_p	$f^n(x_p)$	$f(x_p)$
10^{-2}	5	3.2	5	0.1014	3.1989	3.1989
10^{-4}	8	3.2	6	0.103524	3.1999618	3.1999595
$2.2 * 10^{-6}$	10	3.2	7	0.1035963	3.20000018	3.20000003
10^{-2}	5	4	4	0.8255	4.0006	4.0004
10^{-4}	8	4	5	0.825202	3.9999976	3.9999948
$2.2 * 10^{-6}$	10	4	6	0.8252055	4.00000000	3.99999983
10^{-2}	5	4.25	3	0.9722	4.2490	4.2488
10^{-4}	8	4.25	4	0.972819	4.2500190	4.2500161
$2.2 * 10^{-6}$	10	4.25	5	0.9728105	4.24999999	4.24999981

secant method

With the data above, we have drawn the curves C_γ of length 3.7 with a precision $\varepsilon = 10^{-5}$ for different values of γ.

γ	x
1.2	0.639961
2	0.613581
5	0.552386
12	0.509056

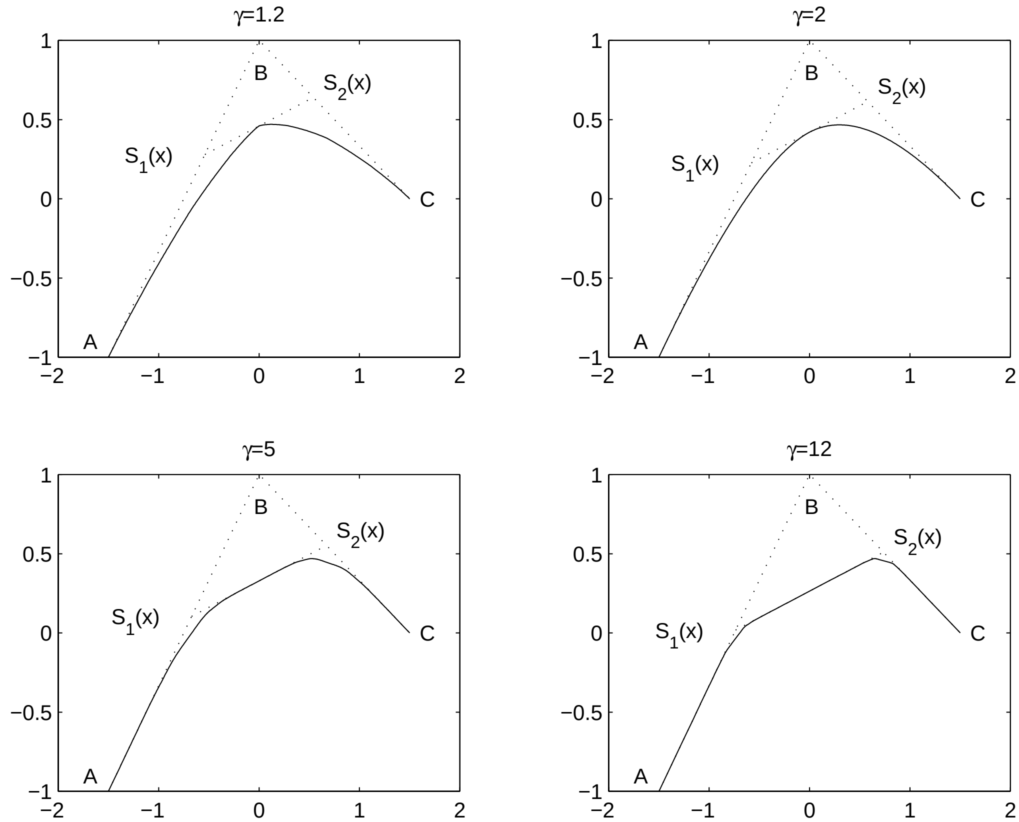

figure 4: The interpolating curves of length 3.7

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