de Rham Transform of a Hermite Subdivision Scheme

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Abstract

For a Hermite subdivision scheme $\mathcal{H}$ of degree d, we define a spectral condition. We have already proved that if it is fulfilled, then there exists an associated affine subdivision scheme $\mathcal{S}$. Moreover, if T is the subdivision matrix of the first difference process $\Delta\mathcal{S}$, if the spectral radius $\rho(T)$ is less than 1, then $\mathcal{S}$ is C^0 and $\mathcal{H}$ is C^d. Generalizing the de Rham corner cutting, from every Hermite subdivision scheme $\mathcal{H}$, we build a new Hermite subdivision scheme, the de Rham transform $\tilde{\mathcal{H}}$. If $\mathcal{H}$ satisfies the spectral condition, then its de Rham transform fulfils it. If $\tilde{S}$ is the associated subdivision scheme to $\tilde{\mathcal{H}}$ and if T and $\tilde{T}$ are the respective subdivision matrices corresponding to $\Delta\mathcal{S}$ and $\Delta\tilde{S}$, then the spectral radii $\rho(T)$ and $\rho(\tilde{T})$ allow the comparison between the indices of smoothness of the limit functions of the schemes $\mathcal{H}$ and $\tilde{\mathcal{H}}$. We apply these results to the Merrien class of Hermite subdivision schemes of degree 1.

1 Introduction

A *Hermite subdivision scheme* $\mathcal{H}$ *of degree* d is a recursive scheme for computing a function $\phi : \mathbb{R} \to \mathbb{R}$ and its d derivatives $\phi', \ldots, \phi^{(d)}$. The initial state of the scheme is a vector function $f_0 : \mathbb{Z} \to \mathbb{R}^{d+1}$. The first component of f_0 is a control value for ϕ, the second component, for ϕ' and so on. The sequence of *refinements* $f_n : \mathbb{Z} \to \mathbb{R}^{d+1}$, $n > 0$, is recursively defined through a family of $(d+1) \times (d+1)$ matrices $\{A(\alpha) = (a_{ij}(\alpha))_{i,j=0,\ldots,d}\}_{\alpha\in\mathbb{Z}}$, a finite number of them being non-zero, by

$$D^{n+1} f_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta) D^n f_n(\beta),\ \alpha \in \mathbb{Z}, n \geq 0, \tag{1}$$

where D is the diagonal matrix with diagonal elements $1, 1/2, \ldots, 1/2^d$.

The family of matrices $\{A(\alpha)\}_{\alpha\in\mathbb{Z}}$ is called the *mask* of the Hermite subdivision scheme $\mathcal{H}$. The *support* of $\mathcal{H}$ is the smallest interval $[\sigma,\sigma']$ containing $\{\alpha\in\mathbb{Z} : A(\alpha)\neq 0\}$.

The outline of the paper is the following: in Section 2, we recall the main results of the previous paper [4] and we define what the spectral condition for a Hermite subdivision scheme $\mathcal{H}$ is. Under this condition, we associate with $\mathcal{H}$ a scalar affine subdivision scheme $\mathcal{S}$. If T is the subdivision matrix of the first difference process $\Delta\mathcal{S}$ and if the spectral radius $\rho(T)$ is less than 1, then $\mathcal{S}$ is C^0 and $\mathcal{H}$ is C^d as shown in [4]. In Section 3, generalizing from de Rham corner cutting [2] and Chaikin algorithm [1], from every Hermite subdivision scheme $\mathcal{H}$, we build a new Hermite subdivision scheme, de Rham transform $\tilde{\mathcal{H}}$. If $\mathcal{H}$ satisfies the spectral condition, then its de Rham transform fulfils it, which insures the existence of the associated scalar affine subdivision scheme $\tilde{\mathcal{S}}$ and of the subdivision matrix $\tilde{T}$ of $\Delta\tilde{\mathcal{S}}$. In Section 3, we compare the indices of smoothness of the limit functions of the schemes $\mathcal{H}$ and $\tilde{\mathcal{H}}$ with the help of the spectral radii $\rho(T)$ and $\rho(\tilde{T})$. In particular, we complete the computation of T and $\tilde{T}$ in the case of any Hermite subdivision scheme belonging to the Merrien class as defined in [7].

2 Associated Vector Subdivision Scheme and Convergence

Definition 1 *We say that a Hermite subdivision scheme of degree d is C^d if for every sequence of refinements $f_n : \mathbb{Z}\to\mathbb{R}^{d+1}$, there exists a C^d-function $\phi:\mathbb{R}\to\mathbb{R}$ for which for every $\epsilon>0$ and for every $L>0$, there exists N such that for every $n>N$, for every $\alpha\in[-L2^n, L2^n]$, $|f_n^{(i)}(\alpha)-\phi^{(i)}(\alpha/2^n)|<\epsilon$, for $i=0,\dots,d$. The function ϕ is called the* limit function *associated to the refinements f_n.*

Definition 2 *A Hermite subdivision scheme of degree d satisfies the* spectral condition *if for $k=0,\dots,d$, there exists a polynomial p_k of degree k such that for all $\alpha\in\mathbb{Z}$,*

$$\sum_{\beta\in\mathbb{Z}} A(\alpha-2\beta)v_k(\beta)=v_k(\alpha)/2^k \text{ where } v_k(\alpha)=(p_k(\alpha),\dots,p_k^{(d)}(\alpha))^T. \quad (2)$$

We use the notation $\mathcal{P}_n$ for the space of polynomials of degree inferior or equal to n.

Lemma 1 *For a given Hermite subdivision scheme of degree d satisfying the spectral condition (2) with polynomials $p_0,p_1,\dots,p_d$, each polynomial p_k of degree k is unique up to a multiplicative factor.*

Proof: Let V the set of vector functions $v : \mathbb{Z} \to \mathbb{C}^{d+1}$ for which there exists a polynomial $p \in \mathcal{P}_d$ such that $v(\alpha) = (p(\alpha), p'(\alpha), \ldots, p^{(d)}(\alpha))^T$ for every $\alpha \in \mathbb{Z}$. The dimension of V is $d+1$: the set $\{v_0, v_1, ..., v_d\}$ is linearly independent as a set of eigenvectors corresponding to distinct eigenvalues, thus it is a basis of V. Moreover, the polynomials $p_0, p_1, ..., p_d$ are linearly independent since their degrees are distinct.

Let $q \in \mathcal{P}_d$ be such that $w(\alpha) = (q(\alpha), q'(\alpha), ..., q^{(d)}(\alpha)^T$ is an eigenvector corresponding to the value $1/2^k$, then $q = \sum_{i=0}^d c_i p_i$ and $w = \sum_{i=0}^d c_i v_i$. If $Hw(\alpha) = \sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta) w(\beta)$, then $w = 2^k H w$ and $\sum_{i=0}^d 2^{k-i} c_i v_i = \sum_{i=0}^d c_i v_i$. If $i \neq k$, then $c_i = 0$ and $w = c_k v_k$. In particular, $q = c_k p_k$. □

Definition 3 *A Hermite subdivision scheme is* interpolatory *if $A(0) = D$ and for all $\alpha \in \mathbb{Z}$ with $\alpha \neq 0$, $A(2\alpha) = 0$. In this case, for $\alpha \in \mathbb{Z}$, $f_n(\alpha) = f_{n+1}(2\alpha)$.*

Lemma 2 *Let $\mathcal{H}$ be an interpolatory Hermite subdivision scheme of degree d which satisfies the spectral condition* (2) *for a polynomial p_k of degree k, $k \in [0, d]$, then $p_k(x)$ is a multiple of x^k.*

Proof: We choose the value $\alpha = 0$ in (2) and we get $Dv_k(0) = v_k(0)/2^k$, i.e. $p_k^{(i)}(0)/2^i = p_k^{(i)}(0)/2^k$ for $i = 0, 1, ..., d$, then $p_k^{(i)}(0) = 0$ for $i \neq k$. From that it follows that $p_k(x) = \sum_{i=0}^d p_k^{(i)}(0) x^i / i! = p_k^{(k)}(0) x^k / k!$. □

A *vector subdivision matrix of order* p is a matrix function $S : \mathbb{Z}^2 \to \mathbb{R}^{p\times p}$ for which there exists an interval $[\sigma, \sigma']$, $\sigma, \sigma' \in \mathbb{Z}$, $\sigma \leq \sigma'$ such that $S(\alpha, \beta) = 0$ whenever $\alpha - 2\beta \notin [\sigma, \sigma']$. If $g_0 : \mathbb{R}^p \to \mathbb{R}$, then the recursive formula $g_{n+1}(\alpha) = \sum_{\beta\in\mathbb{Z}} S(\alpha, \beta) g_n(\beta)$ generates the refinements of g_0 and defines a *vector subdivision scheme* $\mathcal{S}$ *of order* p. The interval $[\sigma, \sigma']$ is a *support* of $\mathcal{S}$. If there exists a matrix function $B : \mathbb{Z} \to \mathbb{R}^{p\times p}$ such that $S(\alpha, \beta) = B(\alpha - 2\beta)$, then B is the *mask* of $\mathcal{S}$. S (and its corresponding vector subdivision scheme) is *affine* if $\sum_{\beta\in\mathbb{Z}} \sum_{j=1}^p s_{ij}(\alpha, \beta) = 1$, $\alpha \in \mathbb{Z}$, $i = 1, 2, ..., p$, where $s_{ij}(\alpha, \beta)$ are the entries of $S(\alpha, \beta)$. The subdivision scheme is *scalar* if $p = 1$. We say that a vector subdivision scheme $\mathcal{S}$ of order p is C^0 if for every sequence of refinements $f_n : \mathbb{Z} \to \mathbb{R}^p$ of $\mathcal{S}$, there is a continuous function $\phi : \mathbb{R} \to \mathbb{R}^p$ such that for every $\epsilon > 0$ and for every $L > 0$, there exists N such that $||f_n(\alpha) - \phi(\alpha/2^n)|| < \epsilon$ for every $n > N$ and every $\alpha \in [-L2^n, L2^n]$. The function ϕ is called the *limit function* associated with the refinements f_n. If for any limit function, all of its components are the same, we say that $\mathcal{S}$ is C^0 *with equal components.*

For $f = (f^{(0)}, \ldots, f^{(d)})^T$ where $f^{(i)} : \mathbb{Z} \to \mathbb{C}$, $i = 0, \ldots, d$, if P_0, P_1 are the respective square matrices of order $d+1$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{(d-2)!}{(d-3)!} & 0 \\ 0 & -\frac{(d-1)!}{0!} & \cdots & -\frac{(d-1)!}{(d-2)!} & 0 \\ -\frac{d!}{0!} & -\frac{d!}{1!} & \cdots & -\frac{d!}{(d-1)!} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1! & 0 \\ \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & (d-1)! & \cdots & 0 & 0 \\ d! & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3}$$

we define the operator Pf

$$Pf(\alpha) = P_0 f(\alpha) + P_1 f(\alpha+1), \alpha \in \mathbb{Z}. \tag{4}$$

Theorem 3 *Let $\mathcal{H}$ be a Hermite subdivision scheme of degree d which satisfies the spectral condition and whose mask $A(\alpha) = (a_{ij}(\alpha))_{i,j=0,\dots,d}$ has support included in $[\sigma, \sigma']$. Let $f_n = (f_n^{(0)}, f_n^{(1)}, \dots, f_n^{(d)})^T$, $n = 0, 1, 2, \dots$ be the refinements of a $\mathcal{H}$. We set $g_n = 2^{nd} P D^n f_n$ and $C(\alpha) = 2^d (P_0 A(\alpha) + P_1 A(\alpha+1))$ where P, P_0, P_1 are defined in (3-4). If we define the matrix function $B(\alpha) = (b_{ij}(\alpha))_{i,j=0,\dots,d}$ by*

$$b_{i,d-j}(\alpha) = \frac{1}{(d-j)!} \sum_{\beta=1}^{\infty} \sum_{k=0}^{j} c_{i,k}(\alpha - 2\beta) \frac{(\beta-1)^{j-k}}{(j-k)!}, \; j = 0, \dots, d-1 \tag{5}$$

$$b_{i,0}(\alpha) = c_{id}(\alpha) \tag{6}$$

for $i = 0, \dots, d$, then the support of B is contained in $[\sigma - 1, \sigma']$ and the sequence g_n is the sequence of refinements of an affine vector subdivision scheme $\mathcal{S}$ whose mask is B.

See [4] for the proof of the previous and the following theorems.

Theorem 4 *Let $\mathcal{H}$ be a Hermite subdivision scheme of degree d which satisfies the spectral condition. Let $B(\alpha) = (b_{ij}(\alpha))_{i,j=0,1,\dots,d}$ be the mask of the Taylor subdivision scheme $\mathcal{S}$ associated with $\mathcal{H}$. We define the functions $s, t : \mathbb{Z}^2 \to \mathbb{R}$*

$$s((d+1)\alpha + i, (d+1)\beta + j) = b_{ij}(\alpha - 2\beta), \quad \alpha, \beta \in \mathbb{Z}, i, j = 0, 1, ..., d, \tag{7}$$

$$t(\alpha, \beta) = -\sum_{\gamma=-\infty}^{\beta} [s(\alpha+1, \gamma) - s(\alpha, \gamma)] = \sum_{\gamma=\beta+1}^{\infty} [s(\alpha+1, \gamma) - s(\alpha, \gamma)]. \tag{8}$$

If, moreover, there is an integer n such that

$$\max\{\sum_{\beta \in \mathbb{Z}} |t_n(\alpha, \beta)| : \alpha \in [0, (d+1)2^n - 1]\} < 1 \tag{9}$$

where $(t_n(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}}$ is the n-th power of the matrix $T = (t(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}}$, then the Taylor subdivision scheme $\mathcal{S}$ is C^0 with equal components and the Hermite subdivision scheme $\mathcal{H}$ is C^d.

Remark 1 *Condition (9) means that $||T^n||_\infty < 1$. As the spectral radius of T is $\rho(T) = \inf\{(||T^n||_\infty)^{1/n} : n \in \mathbb{N}\}$, this condition is equivalent to $\rho(T) < 1$.*

3 de Rham Transform

For a Hermite subdivision scheme $\mathcal{H}$ of degree d, with mask $\{A(\alpha)\}_{\alpha\in\mathbb{Z}}$, let us define a new Hermite subdivision scheme $\tilde{\mathcal{H}}$. From the initial state of the scheme $\tilde{f}_0 : \mathbb{Z} \to \mathbb{R}^{d+1}$, we define the sequence $\tilde{f}_n$ for $n > 0$ by

$$D^{n+1}g(\beta) = \sum_{\gamma\in\mathbb{Z}} A(\beta-2\gamma)D^n\tilde{f}_n(\gamma),\ \beta\in\mathbb{Z} \tag{10}$$

$$D^{n+2}h(\alpha) = \sum_{\beta\in\mathbb{Z}} A(\alpha-2\beta)D^{n+1}g(\beta),\ \alpha\in\mathbb{Z} \tag{11}$$

$$\tilde{f}_{n+1}(\alpha) = h(2\alpha+1),\ \alpha\in\mathbb{Z}. \tag{12}$$

(See Figure 1.)

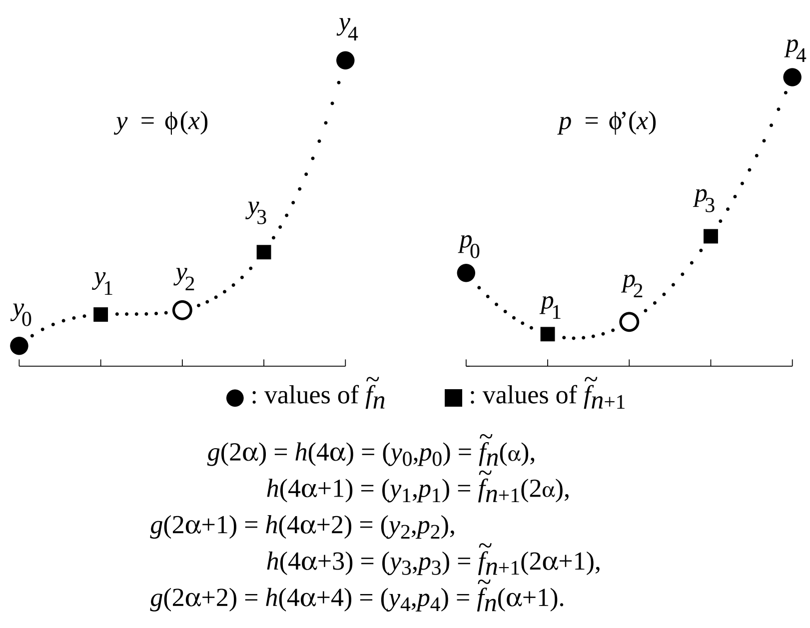

Figure 1: For $d = 1$, the construction of $\tilde{f}_{n+1}$ when $\mathcal{H}$ generates the Hermite cubic interpolant.

Then

$$
\begin{aligned}
& D^{n+1}\tilde{f}_{n+1}(\alpha) = D^{-1}D^{n+2}h(2\alpha+1) \\
= \;& D^{-1}\sum_{\beta\in\mathbb{Z}} A(2\alpha+1-2\beta)\sum_{\gamma\in\mathbb{Z}} A(\beta-2\gamma)D^n\tilde{f}_n(\gamma) \\
= \;& D^{-1}\sum_{\gamma\in\mathbb{Z}}\left(\sum_{\beta\in\mathbb{Z}} A(2\alpha+1-2\beta)A(\beta-2\gamma)\right)D^n\tilde{f}_n(\gamma) \\
= \;& D^{-1}\sum_{\gamma\in\mathbb{Z}}\left(\sum_{\beta\in\mathbb{Z}} A(2\alpha+1-4\gamma-2\beta)A(\beta)\right)D^n\tilde{f}_n(\gamma) \\
= \;& \sum_{\gamma\in\mathbb{Z}} \tilde{A}(\alpha-2\gamma)D^n\tilde{f}_n(\gamma),
\end{aligned}
$$

where

$$\tilde{A}(\alpha) = D^{-1}\sum_{\beta\in\mathbb{Z}} A(2\alpha+1-2\beta)A(\beta), \alpha\in\mathbb{Z}. \qquad (13)$$

Definition 4 *Let $\mathcal{H}$ be a Hermite subdivision scheme with mask $A(\alpha)$. The* de Rham transform $\tilde{\mathcal{H}}$ of $\mathcal{H}$ *is the Hermite subdivision scheme whose mask $\tilde{A}(\alpha)$ is defined by* (13)*.*

If the support of $\mathcal{H}$ is $[\sigma,\sigma']$, then the support of its de Rham transform $\tilde{\mathcal{H}}$ is contained in $[(3\sigma-1)/2,(3\sigma'-1)/2]$.

Theorem 5 *Let $\mathcal{H}$ be a Hermite subdivision scheme of degree d which satisfies the spectral condition* (2)*, then the de Rham transform $\tilde{\mathcal{H}}$ satisfies the corresponding spectral condition $\sum_{\beta\in\mathbb{Z}}\tilde{A}(\alpha-2\beta)\tilde{v}_k(\beta) = \tilde{v}_k(\alpha)/2^k$ where $\tilde{v}_k(\alpha) = (\tilde{p}_k(\alpha),\tilde{p}'_k(\alpha),\ldots,\tilde{p}_k^{(d)}(\alpha))^T$ for an appropriate sequence of polynomials $\tilde{p}_k$ of degree k, $k=0,\ldots,d$.*

Proof: If we use the notations of Definition 2, for $i=0,\ldots,d$, let $v_i(\alpha) = (p_i(\alpha),\ldots,p_i^{(d)}(\alpha))^T$ be the eigenvector of $H = (A(\alpha-2\beta))_{\alpha,\beta\in\mathbb{Z}}$ for the eigenvalue $1/2^i$ where p_i is a polynomial of degree i with its leading coefficient equal to $1/i!$. For $\tilde{H} = (\tilde{A}(\alpha-2\beta))_{\alpha,\beta\in\mathbb{Z}}$, we will build the polynomials $\tilde{p}_k$ of degree k with leading coefficient equal to $1/k!$ and the corresponding eigenvector $\tilde{v}_k(\alpha) = (\tilde{p}_k(\alpha),\ldots,\tilde{p}_k^{(d)}(\alpha))^T$ by a finite recursion.

We begin with a computation to prove that:

$$\sum_{\beta\in\mathbb{Z}}\tilde{A}(\alpha-2\beta)v_k(\beta) = 1/4^k D^{-1}v_k(2\alpha+1),\ k=0,\ldots,d \qquad (14)$$

$$\sum_{\beta\in\mathbb{Z}} \tilde{A}(\alpha-2\beta)v_k(\beta) = D^{-1}\sum_{\beta\in\mathbb{Z}}\sum_{\gamma\in\mathbb{Z}} A(2\alpha-4\beta-2\gamma+1)A(\gamma)v_k(\beta)$$

$$\begin{aligned}
&= D^{-1}\sum_{\beta\in\mathbb{Z}}\sum_{\gamma'\in\mathbb{Z}} A(2\alpha+1-2\gamma')A(\gamma'-2\beta)v_k(\beta) \\
&= D^{-1}\sum_{\gamma'\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} A(2\alpha+1-2\gamma')A(\gamma'-2\beta)v_k(\beta) \\
&= D^{-1}\sum_{\gamma'\in\mathbb{Z}} A(2\alpha+1-2\gamma')1/2^k v_k(\gamma') = 1/4^k D^{-1} v_k(2\alpha+1).
\end{aligned}$$

Step 0. Let $\tilde{p}_0(x) = 1 = p_0(x)$. If $\tilde{v}_0(\alpha) = (\tilde{p}_0(\alpha),\dots,\tilde{p}_0^{(d)}(\alpha))^T$, then $v_0(\alpha) = \tilde{v}_0(\alpha) = (1,0,\dots,0)^T$. With (14), we get

$$\sum_{\beta\in\mathbb{Z}} \tilde{A}(\alpha-2\beta)\tilde{v}_0(\beta) = D^{-1}v_0(2\alpha+1) = v_0(\alpha) = \tilde{v}_0(\alpha)$$

and $\tilde{v}_0(\alpha)$ is an eigenvector of $\tilde{H}$ for the eigenvalue 1.

Step k with $0 < k \le d$. Let us suppose that for $i = 0,\dots,k-1$, $\tilde{v}_i(\alpha) = (\tilde{p}_i(\alpha),\dots,\tilde{p}_i^{(d)}(\alpha))^T$ is an eigenvector of $\tilde{H} = (\tilde{A}(\alpha-2\beta))_{\alpha,\beta\in\mathbb{Z}}$ for the eigenvalue $1/2^i$ where $\tilde{p}_i$ is a polynomial of degree i with its leading coefficient equal to $1/i!$

Since $\{p_k,\tilde{p}_{k-1},\dots,\tilde{p}_0\}$ is a basis of $\mathcal{P}_k$, and since we know that the leading coefficient of p_k is $1/k!$, we can write $p_k(2x+1) = 2^k p_k(x) + \sum_{i=0}^{k-1}\lambda_{k,i}\tilde{p}_i(x)$. When we differentiate up to d times, both sides of the previous equation with respect to x, we get:

$$D^{-1}v_k(2\alpha+1) = 2^k v_k(\alpha) + \sum_{i=0}^{k-1}\lambda_{k,i}\tilde{v}_i(\alpha),\ \alpha\in\mathbb{Z}. \tag{15}$$

Let us define $\tilde{p}_k(x) = p_k(x) + \sum_{i=0}^{k-1}\mu_{k,i}\tilde{p}_i(x)$ where $\mu_{k,i} = -\lambda_{k,i}\frac{2^{i-k}}{2^k-2^i}$ and for $\alpha\in\mathbb{Z}$, let us set $\tilde{v}_k(\alpha) = (\tilde{p}_k(\alpha),\dots,\tilde{p}_k^{(d)}(\alpha))^T$. Firstly, $\tilde{p}_k$ is a polynomial of degree k and its leading coefficient is exactly $1/k!$. Secondly, when we differentiate $\tilde{p}_k$, we obtain:

$$\tilde{v}_k(\alpha) = v_k(\alpha) + \sum_{i=0}^{k-1}\mu_{k,i}\tilde{v}_i(\alpha). \tag{16}$$

Thirdly, for $i = 0,\dots,k-1$, we already know that $\tilde{v}_i(\alpha)$ is an eigenvector of $\tilde{H}$ for the eigenvalue $1/2^i$.

If we set $w_k(\alpha) = \sum_{\beta\in\mathbb{Z}} \tilde{A}(\alpha-2\beta)\tilde{v}_k(\beta)$, with (14), we get:

$$\begin{aligned}
w_k(\alpha) &= 1/4^k D^{-1} v_k(2\alpha+1) + \sum_{i=0}^{k-1} \mu_{k,i}/2^i \tilde{v}_i(\alpha) \\
&= 1/2^k v_k(\alpha) + \sum_{i=0}^{k-1} \left(\lambda_{k,i}/4^k + \mu_{k,i}/2^i\right) \tilde{v}_i(\alpha) \quad \text{(from (15))} \\
&= 1/2^k v_k(\alpha) + \sum_{i=0}^{k-1} \mu_{k,i} \left(-\frac{1}{4^k}\frac{2^k-2^i}{2^{i-k}} + \frac{1}{2^i}\right) \tilde{v}_i(\alpha) \\
&= 1/2^k \left(v_k(\alpha) + \sum_{i=0}^{k-1} \mu_{k,i}\tilde{v}_i(\alpha)\right) = 1/2^k \tilde{v}_k(\alpha) \quad \text{(from (16))}.
\end{aligned}$$

So that $\tilde{v}_k(\alpha)$ is an eigenvector of $\tilde{H}$ for the eigenvalue $1/2^k$. The spectral condition has been proved by induction. □

Corollary 6 *Let $\mathcal{H}$ be an interpolatory scheme of degree d which satisfies the spectral condition, then its de Rham transform $\tilde{\mathcal{H}}$ satisfies the corresponding spectral condition with the sequence of polynomials $\tilde{p}_k(x) = (x-1/2)^k/k!$, $k = 0, \ldots, d$.*

Proof: We use the notations of the previous theorem. From Lemma 1, $\mathcal{H}$ satisfies the spectral condition (2) with the sequence $p_k(x) = x^k/k!$.

For $k = 0$, $p_0(x) = 1$, and we have already seen that $\tilde{p}_0(x) = 1$.

Let us suppose that for $i = 0, \ldots, k-1$, $\tilde{p}_i(x) = (x-1/2)^i/i!$, then

$$\begin{aligned}
p_k(2x+1) &= 2^k x^k/k! + 2^k(x-1/2+1)^k/k! - 2^k(x-1/2+1/2)^k/k! \\
&= 2^k x^k/k! + \frac{2^k}{k!}\sum_{i=0}^{k-1}\binom{k}{i}(1-1/2^{k-i})(x-1/2)^i \\
&= 2^k p_k(x) + \sum_{i=0}^{k-1} \lambda_{k,i}\tilde{p}_i(x), \text{ with } \lambda_{k,i} = \frac{2^k-2^i}{(k-i)!}.
\end{aligned}$$

Since $\mu_{k,i} = -\lambda_{k,i}\frac{2^{i-k}}{2^k-2^i} = -\frac{2^{i-k}}{(k-i)!}$, we obtain

$$\begin{aligned}
\tilde{p}_k(x) &= p_k(x) + \sum_{i=0}^{k-1}\mu_{k,i}\tilde{p}_i(x) = \frac{(x-\frac{1}{2}+\frac{1}{2})^k}{k!} - \sum_{i=0}^{k-1}\frac{2^{i-k}}{(k-i)!}\tilde{p}_i(x) \\
&= \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}2^{i-k}(x-1/2)^i - \frac{1}{k!}\sum_{i=0}^{k-1}\binom{k}{i}(x-1/2)^i 2^{i-k} \\
&= (x-1/2)^k/k!
\end{aligned}$$

and the result is proved by this finite induction. □

4 Examples of Hermite Schemes and their de Rham Transforms

As a model example, we recall the interpolatory Hermite subdivision scheme $\mathcal{H}(\lambda,\mu)$ proposed by Merrien [7]. The non-zero matrices of its mask of are

$$A(-1)=\frac{1}{4}\begin{pmatrix} 2 & 4\lambda \\ 2\mu & 1-\mu \end{pmatrix}, A(0)=D, A(1)=\frac{1}{4}\begin{pmatrix} 2 & -4\lambda \\ -2\mu & 1-\mu \end{pmatrix}.$$

For every choice of λ and μ, $\mathcal{H}(\lambda,\mu)$ reproduces affine functions and satisfies the spectral condition (2) with $p_0(x)=1$, $p_1(x)=x$. Acoording to Theorem 3, an affine vector subdivision scheme $\mathcal{S}$ is associated with $\mathcal{H}(\lambda,\mu)$. The nonzero matrices $B(\alpha)$ of the mask of $\mathcal{S}$ are $B(\alpha)$, $\alpha=-2,...,1$, i.e. $\begin{pmatrix} 0 & 0 \\ 2\lambda & 0 \end{pmatrix}$, $\begin{pmatrix} (1-\mu)/2 & 0 \\ -2\lambda & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -2\lambda & 1 \end{pmatrix}$, $\begin{pmatrix} (1-\mu)/2 & \mu \\ 2\lambda & 1 \end{pmatrix}$.

The subdivision matrix $T=(t(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}}$ of the difference subdivision scheme $\Delta\mathcal{S}$ can be written as follows.

$$(t(\alpha,\beta))_{\alpha\in[0,3],\beta=0,1}=\frac{1}{2}\begin{pmatrix} 2+4\lambda & 4\lambda \\ -1-4\lambda+\mu & 1-4\lambda-\mu \\ 1-4\lambda-\mu & -1-4\lambda+\mu \\ 4\lambda & 2+4\lambda \end{pmatrix}.$$

For every $\alpha\in[0,3]$, $t(\alpha,\beta)=0$ if $\beta\notin[0,1]$. $t(\alpha+4\gamma,\beta+2\gamma)=t(\alpha,\beta)$.

The support of de Rham transform $\tilde{\mathcal{H}}(\lambda,\mu)$ is $[-2,1]$. The respective matrices $\tilde{A}(-2)$, $\tilde{A}(-1)$, $\tilde{A}(0)$, $\tilde{A}(1)$ of its mask are

$$\frac{1}{8}\begin{pmatrix} 2+4\lambda\mu & 6\lambda-2\lambda\mu \\ 6\mu-2\mu^2 & (1-\mu)^2+8\lambda\mu \end{pmatrix}, \frac{1}{8}\begin{pmatrix} 6-4\lambda\mu & 6\lambda+2\lambda\mu \\ 6\mu-2\mu^2 & 3-4\mu+\mu^2-8\lambda\mu \end{pmatrix},$$
$$\frac{1}{8}\begin{pmatrix} 6-4\lambda\mu & -6\lambda-2\lambda\mu \\ -6\mu+2\mu^2 & 3-4\mu+\mu^2-8\lambda\mu \end{pmatrix}, \frac{1}{8}\begin{pmatrix} 2+4\lambda\mu & -6\lambda+2\lambda\mu \\ -6\mu+2\mu^2 & (1-\mu)^2+8\lambda\mu \end{pmatrix}.$$

$\tilde{\mathcal{H}}(\lambda,\mu)$ satisfies the spectral condition with $\tilde{p}_0(x)=1$, $\tilde{p}_1(x)=x-1/2$. From Theorem 3, an affine vector subdivision scheme $\tilde{\mathcal{S}}$ is associated with $\tilde{\mathcal{H}}(\lambda,\mu)$. The nonzero matrices $\tilde{B}(\alpha)$ of the mask of $\tilde{\mathcal{S}}$ are $\tilde{B}(\alpha)$, $\alpha\in[-3,1]$. Using Theorem 4, we define two infinite matrices $\tilde{S}=(\tilde{s}(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}}$ and $\tilde{T}=(\tilde{t}(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}}$. For $\alpha,\beta\in\mathbb{Z}$, for $i,j=0,1$, $\tilde{s}(2\alpha+i,2\beta+j)=\tilde{b}_{ij}(\alpha,\beta)$ and $\tilde{t}(\alpha,\beta)=-\sum_{\gamma=-\infty}^{\beta}[\tilde{s}(\alpha+1,\gamma)-\tilde{s}(\alpha,\gamma)]$.

After computation, we obtain the following information about $\tilde{T}$. The entries

$\tilde{t}(\alpha,\beta)$ are 0 if $\alpha = [0,3]$ and $\beta \notin [0,3]$. The matrix $(\tilde{t}(\alpha,\beta))_{\alpha,\beta\in[0,3]}$ is

$$\frac{1}{4}\begin{pmatrix} w_1 & w_2 & 0 & 0 \\ w_2 & w_1 & 0 & 0 \\ w_3 & w_4 & w_5 & w_6 \\ w_6 & w_5 & w_4 & w_3 \end{pmatrix}$$

where $w_1 = 3 - 4\mu + \mu^2 - 12\lambda\mu$, $w_2 = -(1-\mu)^2 - 4\lambda\mu$, $w_3 = (1-\mu)^2 - 6\lambda + 10\lambda\mu$, $w_4 = -1 + 4\mu - \mu^2 - 6\lambda + 6\lambda\mu$, $w_5 = 2 + 6\lambda + 2\lambda\mu$, $w_6 = 6\lambda - 2\lambda\mu$. Moreover, $\tilde{T} = (t(\alpha,\beta))$ is periodic: $\tilde{t}(\alpha+4, \beta+2) = t(\alpha,\beta)$ for $\alpha, \beta \in \mathbb{Z}$. Consequently, $\tilde{T}$ is entirely known.

Computing the spectral radii $\rho(T)$ and $\rho(\tilde{T})$ enables us to compare the two Hermite subdivision schemes, $\mathcal{H}(\lambda,\mu)$ and $\tilde{\mathcal{H}}(\lambda,\mu)$.

If $\rho(T) < 1$, then $\mathcal{H}(\lambda,\mu)$ is C^1 and any limit function ϕ of $\mathcal{H}(\lambda,\mu)$ has the property that $\phi'(x) - \phi'(x+h) = O(h^p)$ as $h \to 0$ for any $p < -\log(\rho(T)/\log 2$. We have similar properties for $\tilde{\mathcal{H}}(\lambda,\mu)$ and the Hermite subdivision scheme $\tilde{\mathcal{H}}(\lambda,\mu)$ generates smoother limits than $\mathcal{H}(\lambda,\mu)$ if $\rho(\tilde{T}) < \rho(T)$.

The numbers $\rho(T)$ and $\rho(\tilde{T})$ are not easy to compute, but they admit similar upper bounds

$$\rho(T) \le ||T^n||_\infty^{1/n}, \quad \rho(\tilde{T}) \le ||\tilde{T}^n||_\infty^{1/n} \tag{17}$$

for every $n > 0$. Lower bounds for these spectral radii follow from the following lemma.

Lemma 7 *Let $T = (t(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}}$ and $J \subset \mathbb{Z}$ be such that $t(\alpha,\beta) = 0$ for every $\alpha \in J$ and for every $\beta \notin J$ and let $[T]$ be the matrix $(t(\alpha,\beta))_{\alpha,\beta\in J}$, then the spectral radius of $[T]$ is bounded by the spectral radius of T.*

Proof: For any matrix $U = (u(\alpha,\beta))_{\alpha,\beta\in\mathbb{Z}}$, we set $[U] = (u(\alpha,\beta))_{\alpha,\beta\in J}$. By induction on n, we get $[T^n] = [T]^n$. It follows that $||\,[T]^n||_\infty \le ||T^n||_\infty$ and

$$\rho([T]) = \inf\{||[T]^n||_\infty^{1/n} : n > 0\} \le \inf\{||T^n||_\infty^{1/n} : n > 0\} = \rho(T).$$

□

Let T be the subdivision matrix of the difference subdivision scheme $\Delta\mathcal{S}$ and let $[T]$ be the matrix $(t(\alpha,\beta))_{\alpha,\beta\in[0,1]}$, then from Lemma 7, $\rho([T]$ is a lower bound of $\rho(T)$. Similarly, let $\tilde{T}$ be the subdivision matrix of the difference subdivision scheme $\Delta\tilde{\mathcal{S}}$ and let $[\tilde{T}]$ be the matrix $(\tilde{t}(\alpha,\beta))_{\alpha,\beta\in[0,3]}$, then $\rho([\tilde{T}]$ is a lower bound of $\rho(\tilde{T})$. The roots of the characteristic polynomial of $[\tilde{T}]$ are $w_1 \pm w_2$ and the two roots of the quadratic polynomial: $z^2 - (w_3 + w_5)z + w_3w_5 - w_4w_6$. Thus

$$\rho([\tilde{T}]) = \max(|w_1| + |w_2|, |z_1|, |z_2|)$$

where z_1, z_2 are the roots of the previous quadratic polynomial.

In the following table, we give the results of the computations of the spectral radii for 4 choices of (λ, μ). The upper bounds for the first two choices were obtained from (17) with $n = 30$.

λ	μ	$\rho(T)$	$\rho(\tilde{T})$
1/10	3/2	1.21	$\in [0.85, 0.88]$
1/36	9/10	1.05	$\in [0.57, 0.59]$
-1/8	3/2	0.50	0.50
-1/8	3	0.50	1.75

Let $\Omega = \{(\lambda, \mu) : \mathcal{H}(\lambda, \mu) \text{ is } C^1\}$ and $\tilde{\Omega} = \{(\lambda, \mu) : \tilde{\mathcal{H}}(\lambda, \mu) \text{ is } C^1\}$. In Fig. 2, we compare Ω and $\tilde{\Omega}$. According to Theorem 18 of [3], $\Omega = \{(\lambda, \mu) : \lambda \in (-1/2, 0), \mu \in (0, \min(-1/2\lambda), 3/(1 + 2\lambda))\}$; in the figure, Ω is the region bounded by the dashed curve and the segment $(-1/2, 0) \times \{0\}$. The black area in the figure is the region $\{(\lambda, \mu) : ||\tilde{T}^{20}|| < 1\}$. The set $\{(\lambda, \mu) : \rho([\tilde{T}]) < 1\}$ is the union of the black area and the gray area. There is no guarantee that every gray point belongs to $\tilde{\Omega}$. But from Theorem 4,and the previous lemma,

$$\{(\lambda, \mu) : ||\tilde{T}^{20}|| < 1\} \subset \tilde{\Omega} \subset \{(\lambda, \mu) : \rho([\tilde{T}]) < 1\}.$$

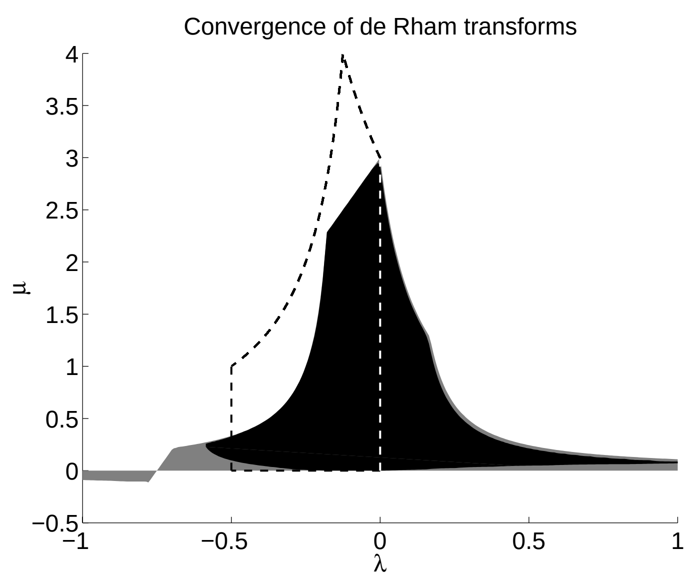

Figure 2: Comparison between the region of convergence of $\mathcal{H}(\lambda, \mu)$ and that of $\tilde{\mathcal{H}}(\lambda, \mu)$.

From Fig. 2, it is obvious that $\Omega \setminus \tilde{\Omega} \neq \emptyset$ and $\tilde{\Omega} \setminus \Omega \neq \emptyset$, which corroborates the previous table.

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