A family of Hermite interpolants by bisection algorithms

J.L. Merrien

Laboratoire L.A.N.S., I.N.S.A. de Rennes, 20 Avenue des Buttes de Coësmes, 35043-Rennes cedex, France

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A two point subdivision scheme with two parameters is proposed to draw curves corresponding to functions that satisfy Hermite conditions on $[a, b]$. We build two functions f and f^1 on dyadic numbers and for some values of the parameters, f is in $\mathscr{C}^1$ with $f^1 = f'$. Examples are provided which show how different the curves can be.

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1. Introduction

Suppose that we know the values of a function and its first derivatives at two real distinct points a and b, i.e.: $f(a)$, $f^1(a)$, $f(b)$, $f^1(b)$. We build $f(i)$ and $f^1(i)$ where i is the middle point of $[a, b]$. Then we proceed to obtain two functions f and f^1 defined on the dyadic numbers $D = \bigcup_0^\infty D_n$, where:

$$D_n = \left\{ x_n^k = a + k\frac{(b-a)}{2^n},\ k = 0 \text{ to } 2^n \right\}.$$

As D is dense in $[a, b]$, we get a $\mathscr{C}^1$ interpolant as soon as we can extend f and f^1 to $[a, b]$ by continuity and prove that $f' = f^1$. This process defines a general family of interpolants containing Hermite cubics and quadratic splines as particular cases. The idea is to get an interpolant without solving any system and to extend these algorithms in two dimensions on triangles for which we usually need polynomials of high degrees. As in Dyn et al. [1] the construction is local and needs four data (two points instead of four, but the derivatives on those points are also necessary to get the interpolant). The construction stops as soon as we reach the screen limits.

2. Definition and construction

EXAMPLE 1

Evaluating the Hermite cubic interpolant at the middle point $m=(a+b)/2$, we get:

$$f(m)=\frac{1}{2}[f(b)+f(a)]-\frac{h}{8}[f^1(b)-f^1(a)],$$

$$f^1(m)=\frac{3}{2h}[f(b)-f(a)]-\frac{1}{4}[f^1(b)+f^1(a)],$$

where $h=b-a$ and $f^1=f'$.

By reiterating the process, and through unicity of this interpolant, we obtain the same form for m_1, the middle point of $[a, m]$ with h replaced by $h/2$ and b by $m\ldots$, so that the process converges to the polynomial.

EXAMPLE 2

Quadratic spline: using the same pattern for the Hermite quadratic spline with one knot at m, we get:

$$f(m)=\frac{1}{2}[f(b)+f(a)]-\frac{h}{8}[f^1(b)-f^1(a)],$$

$$f^1(m)=\frac{2}{h}[f(b)-f(a)]-\frac{1}{2}[f^1(b)+f^1(a)].$$

CONSTRUCTION AND ALGORITHM

Step 0: at the two points $a=x_0^0$ and $b=x_0^1$ we know f and f^1 and $h_0=b-a$.
Step 1: setting $x_1^0=x_0^0$, $x_1^1=(x_0^0+x_0^1)/2$, $x_1^2=x_0^1$, we define:

$$f(x_1^1)=\lambda[f(x_0^1)+f(x_0^0)]+\lambda' h_0[f^1(x_0^1)-f^1(x_0^0)], \tag{2.1}$$

$$f^1(x_1^1)=\frac{\mu}{h_0}[f(x_0^1)-f(x_0^0)]+\mu'[f^1(x_0^1)+f^1(x_0^0)], \tag{2.2}$$

$$h_1=h_0/2.$$

Step $n+1$: we already know f and f^1 on D_n, so that we only have to define $f(x_{n+1}^k)$ and $f^1(x_{n+1}^k)$ for k odd:

$$f(x_{n+1}^{2p+1})=\lambda[f(x_n^{p+1})+f(x_n^p)]+\lambda' h_n[f^1(x_n^{p+1})-f^1(x_n^p)],$$

$$f^1(x_{n+1}^{2p+1})=\frac{\mu}{h_n}[f(x_n^{p+1})-f(x_n^p)]+\mu'[f^1(x_n^{p+1})+f^1(x_n^p)],$$

$$h_{n+1}=h_n/2=(b-a)/2^{n+1}.$$

Now f and f^1 have been defined on $D=\bigcup_0^\infty D_n$. We shall first study their continuity on D. As it is not sufficient to extend them to $[a, b]$, we shall then see when they are uniformly continuous on D.

DEFINITION
(f, f^1) is a $\mathscr{C}^1$ interpolant on a set A if f is continuous and admits a first continuous derivative f' with $f'=f^1$.

3. First conditions for $\mathscr{C}^1$ interpolation on D

PROPOSITION 1
If (f, f^1) is a $\mathscr{C}^1$ interpolant on D, then $\lambda=1/2$ and $\mu+2\mu'=1$.

Proof
On step $n+1$

$$f\left(a+\frac{h_n}{2}\right)=\lambda\left[f(a+h_n)+f(a)\right]+\lambda' h_n\left[f'(a+h_n)-f'(a)\right],$$

$$f'\left(a+\frac{h_n}{2}\right)=\frac{\mu}{h_n}\left[f(a+h_n)-f(a)\right]+\mu'\left[f'(a+h_n)+f'(a)\right].$$

Using Taylor expansions of f and f^1 we get:

$$f(a)+\frac{h_n}{2}f'(a)=\lambda\left[2f(a)+h_n f'(a)\right]+o(h_n),$$

$$f'(a)=\mu\left[f'(a)\right]+\mu'\left[2f'(a)\right]+o(h_n),$$

so when n goes to $+\infty$, we obtain $\lambda=1/2$ and $\mu+2\mu'=1$. □

From now on, we will suppose that the parameters used in our construction verify the conditions of proposition 1: $\lambda=1/2$, $\mu'=(1-\mu)/2$.

Remarks
(1) Likewise if we want the extension of f to be in $\mathscr{C}^2[a, b]$, we get $\lambda'=-1/8$ and for f extended in $\mathscr{C}^3[a, b]$, $\lambda'=-1/8$, $\mu=3/2$.
(2) The two examples given in section 2 verify the conditions of proposition 1.
(3) Note that if initial conditions are: $f^1(a)=f^1(b)$ and $f(b)=f(a)+(b-a)f^1(a)$, the algorithm converges to the function f defined by $f(x)=f(a)+(x-a)f^1(a)$ for any parameters λ' and μ.

PROPOSITION 2
f^1 is continuous on D and f admits a derivative f' with $f'=f^1$ on D (i.e., (f, f^1) is a $\mathscr{C}^1$ interpolant on D) if and only if:

$$|3-\mu\pm\delta|<4 \quad \text{with} \quad \delta=\left[(\mu+1)^2+32\lambda'\mu\right]^{1/2}\in\mathbb{C}.$$

Proof

Because the construction is symmetrical, it is sufficient to study the property at the right side of a point. As the algorithm is local (i.e. to study what happens between x_n^k and x_n^{k+1}, we can reinitialize the construction on x_n^k and x_n^{k+1}), the study at the left bound a is sufficient. So let:

$$U_n = \begin{pmatrix} \dfrac{f(a+h_n)-f(a)}{h_n} - f^1(a) \\ f^1(a+h_n)-f^1(a) \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

$$x_{n+1} = \frac{1}{h_{n+1}}\left(\tfrac{1}{2}[f(a+h_n)-f(a)] + \lambda' h_n[f^1(a+h_n)-f^1(a)]\right) - f^1(a)$$

$$= \frac{1}{h_n}[f(a+h_n)-f(a)] - f^1(a) + 2\lambda'[f^1(a+h_n)-f^1(a)]$$

$$= x_n + 2\lambda' y_n \quad \text{as } h_{n+1} = \frac{h_n}{2},$$

$$y_{n+1} = \frac{\mu}{h_n}[f(a+h_n)-f(a)] + \frac{1-\mu}{2}[f^1(a+h_n)+f^1(a)] - f^1(a)$$

$$= \mu\left(\frac{1}{h_n}[f(a+h_n)-f(a)] - f^1(a)\right) + \frac{1-\mu}{2}[f^1(a+h_n)-f^1(a)],$$

so that

$$U_{n+1} = MU_n,$$

where

$$M = \begin{pmatrix} 1 & 2\lambda' \\ \mu & \dfrac{1-\mu}{2} \end{pmatrix}.$$

f and f^1 have the required properties if and only if U_n tends to 0 as n tends to $+\infty$, i.e. if and only if $\rho(M) < 1$ where $\rho(M)$ is the spectral radius of M.

$$\det|M - XI| = X^2 - \frac{3-\mu}{2}X + \frac{1-\mu}{2} - 2\lambda'\mu.$$

The discriminant is

$$\Delta = \frac{(3-\mu)^2}{4} - 2(1-\mu) + 8\lambda'\mu = \frac{(1+\mu)^2 + 32\lambda'\mu}{4}.$$

So we have the desired property. The domain is given in fig. 1. □

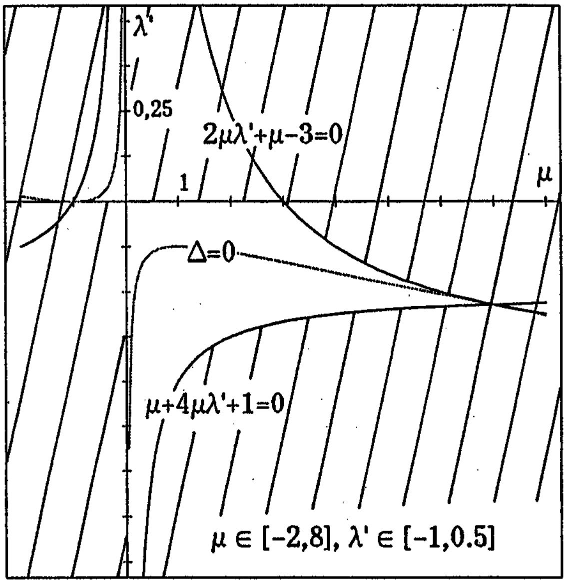


Fig. 1

Remarks

(1) $(\mu+1)^2+32\lambda'\mu$ may be negative.

(2) For Hermite cubic, we have:

$$\lambda' = -\frac{1}{8}, \quad \mu = \frac{3}{2} \quad \text{and} \quad \Delta = \frac{1}{16},$$

so that

$$\lambda_1 = \frac{3-\mu+\delta}{4} = \frac{1}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\mu-\delta}{4} = \frac{1}{4}.$$

For quadratic splines:

$$\lambda' = \frac{1}{8}, \; \mu = 2 \quad \text{and} \quad \Delta = \frac{1}{4},$$

so that

$$\lambda_1 = \frac{3-\mu+\delta}{4} = \frac{1}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\mu-\delta}{4} = 0.$$

(3) Unfortunately, the above conditions are not sufficient to extend f and f^1 to $[a, b]$. For example, for $\mu = +3$ and $\lambda' = -1/6$, which is a point of the domain, (the eigenvalues are both 0), we have drawn f and f^1 (see fig. 2). f^1 is discontinuous. Initial values are $f(0) = f'(0) = f(1) - 2 = f'(1) = 0$.

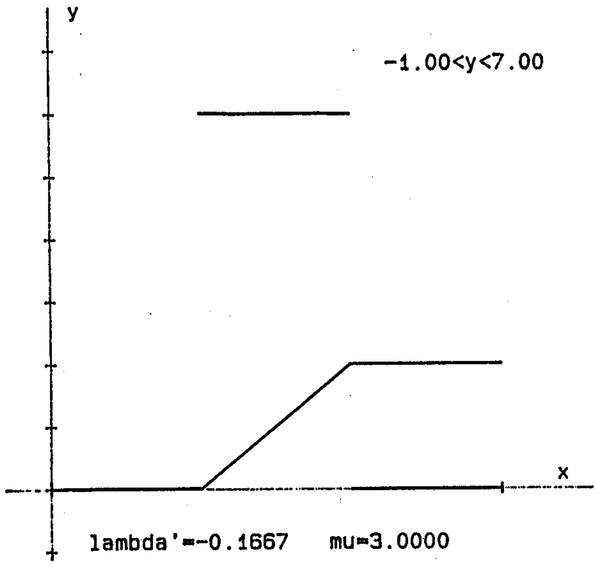


Fig. 2

4. Conditions for the convergence of the algorithm

We now want to know when f and f^1 are uniformly continuous on D, in which case we can extend them by continuity on $[a, b]$. Let $F(a, b)$ be defined by:

$$F(a, b)=\begin{pmatrix} [f(b)+f(a)]/2 \\ [f(b)-f(a)]/2 \\ [f^1(b)-f^1(a)]h \\ [f^1(b)+f^1(a)]h \end{pmatrix},$$

where $h=b-a$.

Using formulas (2.1) and (2.2) with $\lambda=1/2$ and $\mu'=(1-\mu)/2$, we get $F(a, m)=A_1F(a, b)$ and $F(m, b)=A_2F(a, b)$, where m is the middle point of $[a, b]$ and

$$A_i=\begin{pmatrix} 1 & \frac{\epsilon_i}{2} & \frac{\lambda'}{2} & 0 \\ 0 & \frac{1}{2} & -\epsilon_i\frac{\lambda'}{2} & 0 \\ 0 & -\epsilon_i\mu & \frac{1}{4} & \epsilon_i\frac{\mu}{4} \\ 0 & \mu & \frac{\epsilon_i}{4} & \frac{2-\mu}{4} \end{pmatrix}, \quad \epsilon_1=-1 \text{ and } \epsilon_2=1.$$

Then, we have $F(x_n^k, x_n^{k+1}) = A_{\alpha_1} \times A_{\alpha_2} \times \ldots \times A_{\alpha_n} F(a, b)$ where $\alpha_k \in \{1, 2\}$ for $k = 1$ to n. Let $B_i = H^{-1} A_i H$ with:

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix}$$

obtained when studying the eigenvalues of A_i. We get

$$B_i = \begin{pmatrix} 1 & \frac{\epsilon_i}{2} & \frac{\lambda'}{2} & 0 \\ 0 & \frac{1}{2} & -\epsilon_i \frac{\lambda'}{2} & 0 \\ 0 & 0 & \frac{1}{4} & \epsilon_i \frac{\mu}{4} \\ 0 & 0 & \epsilon_i \frac{8\lambda' + 1}{4} & \frac{2 - \mu}{4} \end{pmatrix}$$

Setting $G(x_n^k, x_n^{k+1}) = H^{-1} F(x_n^k, x_n^{k+1})$, we have:

$$G(x_n^k, x_n^{k+1}) = \begin{pmatrix} \left[f(x_n^{k+1}) + f(x_n^k)\right]/2 \\ \left[f(x_n^{k+1}) - f(x_n^k)\right]/2 \\ \left[f^1(x_n^{k+1}) - f^1(x_n^k)\right] h_n \\ \left[f^1(x_n^{k+1}) + f^1(x_n^k)\right] h_n - 2\left[f(x_n^{k+1}) - f(x_n^k)\right] \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and

$$G(x_n^k, x_n^{k+1}) = B_{\alpha_1} \ldots B_{\alpha_n} G(a, b).$$

Let $\| \ \|_n$ denote both a norm in $\mathbb{R}^n$ and the associated norm on the set of square matrices, $\mathbb{R}^{n,n}$.

LEMMA 3

Suppose we have a sequence of matrices M_j of the form

$$M_j = \begin{pmatrix} k_j & m_j^{\mathrm{T}} \\ 0 & N_j \end{pmatrix},$$

where $k_j \in \mathbb{R}$, $m_j \in \mathbb{R}^{n-1}$, $N_j \in \mathbb{R}^{n-1,n-1}$. If there are constants c, d and e so that for each j: $\| N_j \|_{n-1} \leqslant c$, $|k_j| < d$ and $\| m_j \|_{n-1} \leqslant e$, then there is a constant C' independent of p, c and d for which we have:

$$\| M_1 \times \ldots \times M_p \|_n \leqslant C' \left(\sum_{i=0}^{p-1} d^i c^{p-1-i} + d^p + c^p \right) \quad \text{for any } p \geqslant 1.$$

Proof

All the norms are equivalent on $\mathbb{R}^n$, so that we can change the norm by multiplication by a constant. As the Euclidian norm verifies the Cauchy–Schwarz inequality, for any norm of $\mathbb{R}^{n-1}$ we get:

$$|u^{\mathrm{T}}v| \leqslant C_1 \|u\|_{n-1} \times \|v\|_{n-1}.$$

Now

$$M_1 \times \ldots \times M_p = R_p = \begin{pmatrix} k_1 \ldots k_p & r_p^{\mathrm{T}} \\ 0 & N_1 \ldots N_p \end{pmatrix}.$$

$$r_p^{\mathrm{T}} = k_1 \ldots k_{p-1} m_p^{\mathrm{T}} + \sum_{i=1}^{p-2} k_1 \ldots k_i m_{i+1}^{\mathrm{T}} N_{i+2} \ldots N_p + m_1^{\mathrm{T}} N_2 \ldots N_p.$$

So

$$\|r_p\|_{n-1} \leqslant d^{p-1}e + \sum_{i=1}^{p-2} d^i C_1 e c^{p-i-1} + C_1 e c^{p-1}$$
$$\leqslant C_2 \left(\sum_{i=0}^{p-1} d^i c^{p-i+1} \right).$$

As $|k_1 \ldots k_p| \leqslant d^p$ and $\|N_1 \ldots N_p\|_{n-1} \leqslant c^p$, we get:

$$\|M_1 \times \ldots \times M_p\|_n \leqslant C' \left(\sum_{i=0}^{p-1} d^i c^{p-1-i} + d^p + c^p \right). \quad \square$$

So, let us use again B_i and write:

$$B_i = \begin{pmatrix} 1 & b_i^{\mathrm{T}} \\ 0 & B_i' \end{pmatrix} \quad \text{and} \quad B_i' = \begin{pmatrix} \frac{1}{2} & b_i'^{\mathrm{T}} \\ 0 & B_i'' \end{pmatrix},$$

where

$$B_i'' = \begin{pmatrix} \dfrac{1}{4} & \dfrac{\epsilon_i \mu}{4} \\ \epsilon_i \dfrac{8\lambda' + 1}{4} & \dfrac{2-\mu}{4} \end{pmatrix},$$

and let $C_i = 4B_i''$.

LEMMA 4

If there is a norm $\| \;\; \|$ on $\mathbb{R}^2$ such that:

(1) $\|C_i\| \leqslant 1$ for $i = 1, 2$, then f and f^1 are Lipschitz.

(2) $1 < \|C_i\| < k < 2$, then f is Lipschitz, f^1 is Hölder and for x and y in D:

$$|f^1(x) - f^1(y)| \leqslant K|x-y|^{-\ln(k/2)/\ln 2}.$$

In both cases $f' = f^1$.

Proof

In the first case $\| B''_{\alpha_1} \dots B''_{\alpha_n} \| \leqslant (1/4)^n$ so that $| f^1(x_n^{k+1}) - f^1(x_n^k) | \times h_n \leqslant (1/4)^n$. Let us recall that $h_n = h/2^n = x_n^{k+1} - x_n^k$ so that $| f^1(x_n^{k+1}) - f^1(x_n^k) | \leqslant k_1 | x_n^{k+1} - x_n^k |$. By lemma 3,

$$\| B'_{\alpha_1} \dots B'_{\alpha_n} \| \leqslant k_2 \left[\sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i \left(\frac{1}{4}\right)^{n-1-i} + \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n \right]$$

$$\leqslant k_2 \left(\frac{1}{2}\right)^n \left(2 \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^{n-1-i} + 1 + \left(\frac{1}{2}\right)^n \right)$$

$$\leqslant k_3 \left(\frac{1}{2}\right)^n .$$

So we get $| f(x_n^{k+1}) - f(x_n^k) | \leqslant k | x_n^{k+1} - x_n^k |$.

The same method is used in the second case.

Now $| [f^1(x_n^{k+1}) + f^1(x_n^k)] h_n - 2[f(x_n^{k+1}) - f(x_n^k)] | \leqslant C^n$, with $C < 1/2$, so that

$$\left| \frac{f(x_n^{k+1}) - f(x_n^k)}{h_n} - [f^1(x_n^{k+1}) + f(x_n^k)] \right| \leqslant \frac{2^{n-1} C^n}{h} .$$

As f^1 is continuous,

$$\lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} - f^1(x) \right| = 0 \quad \text{for } x \text{ and } x+h \text{ in } D. \quad \square$$

We can now study the conditions on λ' and μ to get a convergence of the algorithm to an interpolant on $[a, b]$.

Instead of studying B''_i or C_i, we change our matrix into $D_i = H'^{-1} C_i H'$, where

$$H' = \begin{pmatrix} 1 & 0 \\ 0 & -1/\mu \end{pmatrix}.$$

Note that if $\mu = 0$, f^1 is a linear function on $[a, b]$ (see (2.2)) and (f, f^1) will not be an interpolant for some initial conditions. We get:

$$D_1 = \begin{pmatrix} 1 & 1 \\ -a & 1-b \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & -1 \\ a & 1-b \end{pmatrix},$$

$$a = -\mu(1 + 8\lambda') \quad \text{and} \quad b = \mu - 1.$$

(1) f^1 Lipschitz: Suppose (f, f^1) is an interpolant and $\rho(D_1) = \rho(C_1) > 1$. Then, either

$$| [f^1(a + h_n) - f^1(a)] \times h_n | > k_4 \left(\frac{\rho(C_1)}{4} \right)^n$$

$$\Leftrightarrow \left| \frac{f^1(a + h_n) - f^1(a)}{h_n} \right| > k_5 \rho(C_1)^n,$$

or

$$|[f^1(a+h_n)+f^1(a)]\times h_n-2[f(a+h_n)-f(a)]|>k_4\left(\frac{\rho(C_1)}{4}\right)^n$$

$$\Leftrightarrow\left|\frac{f^1(a+h_n)-f^1(a+\theta_n h_n)}{(1-\theta_n)h_n}\cdot(1-\theta_n)+\frac{f^1(a)-f^1(a+\theta_n' h_n)}{\theta_n' h_n}\theta_n'\right|$$

$$>k_4\rho(C_1)^n,$$

with θ_n and θ_n' in]0, 1[.

Both cases are impossible if f^1 is Lipschitz. Therefore we have a necessary condition.

Now let us denote by S and P the sum and product of the eigenvalues of D_1 (or D_2) and $\bar{S}$ and $\bar{P}$ for D_1D_2. To obtain $\rho(D_1)\leqslant 1$ and $\rho(D_2)\leqslant 1$, we must have:

$$\begin{cases}1-S+P\geqslant 0,\\ P\leqslant 1,\\ 1-\bar{S}+\bar{P}\geqslant 0.\end{cases}$$

As $S=2-b$, $P=1+a-b$, $\bar{S}=1+2a+(1-b)^2$, $\bar{P}=(1+a-b)^2$ we get:

$$\begin{cases}a\geqslant 0\\ a-b\leqslant 0\\ -2a-(1-b)^2+(1+a-b)^2\geqslant 0\end{cases}\Leftrightarrow\begin{cases}a\geqslant 0\\ b-a\geqslant 0\\ a(a-2b)\geqslant 0\end{cases}.$$

Necessarily $a=0$. So

$$D_1=\begin{pmatrix}1 & 1\\ 0 & 1-b\end{pmatrix}\quad\text{and}\quad D_2=\begin{pmatrix}1 & -1\\ 0 & 1-b\end{pmatrix}.$$

To get $\rho(D_i)\leqslant 1$, we must put $|1-b|\leqslant 1$.

Conversely if $|1-b|<1$, by lemma 3, we easily obtain: $\|D_{\alpha_1}\dots D_{\alpha_n}\|\leqslant k_5$, so $\|C_{\alpha_1}\dots C_{\alpha_n}\|\leqslant k_6$, k_6 independent of n.

If we suppose $1-b=1$, then

$$D_1^n=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix},$$

so that we will not have f^1 Lipschitz as above. Same if $1-b=-1$,

$$(D_1D_2)^n=\begin{pmatrix}1 & -n\\ 0 & +1\end{pmatrix}.\quad\square$$

So we have a necessary and sufficient condition to get f^1 Lipschitz.

PROPOSITION 5

(f, f^1) is a $\mathscr{C}^1$ interpolant with f^1 Lipschitz if and only if

$\lambda'=-1/8$ and $|2-\mu|<1$.

Remark

For Hermite cubic, $\lambda' = -1/8$, $\mu = 3/2$ and for quadratic spline, $\lambda' = -1/8$, $\mu = 2$; we are in the case of f^1 Lipschitz.

(2) f^1 continuous. Writing $\rho(D_1) \leqslant 2$ gives us the same conditions we had in proposition 2. This is not sufficient. We use again S, $P\ldots$, defined above. As we need $\rho(D_1) \leqslant 2$ and $\rho(D_1D_2) \leqslant 4$, we get:

$$(*)\begin{cases} P \leqslant 4 \\ 4-2S+P \geqslant 0 \\ 4+2S+P \geqslant 0 \\ 16-4\bar{S}+\bar{P} \geqslant 0 \end{cases} \Leftrightarrow \begin{cases} 3-a+b \geqslant 0 \\ +b-a+1 \geqslant 0 \\ 9-3b+a \geqslant 0 \\ 12-8a-4(1-b)^2+(1+a-b)^2 \geqslant 0; \end{cases}$$

then $12-8a-b(1-b)^2+(a+1-b)^2 = (a-3b-3)(a+b-3)$.

$$(*) \Leftrightarrow \begin{cases} 3-a+b \geqslant 0 \\ 1+a+b \geqslant 0 \\ 9+a-3b \geqslant 0 \\ (3-a+3b)(3-a-b) \geqslant 0. \end{cases}$$

We now have new necessary conditions:

$$\begin{cases} |3-\mu \pm \delta| < 4, \\ \mu > 0, \\ \lambda' \geqslant -1/2, \\ 2\mu\lambda'+1 \geqslant 0. \end{cases}$$

Conversely, let us use the norm $\|(x, y)\| = |x| + k|y|$ on $\mathbb{R}^2$, with $k = 1/(|a|+\epsilon)$, where ϵ is to be chosen in $\mathbb{R}_+^*$:

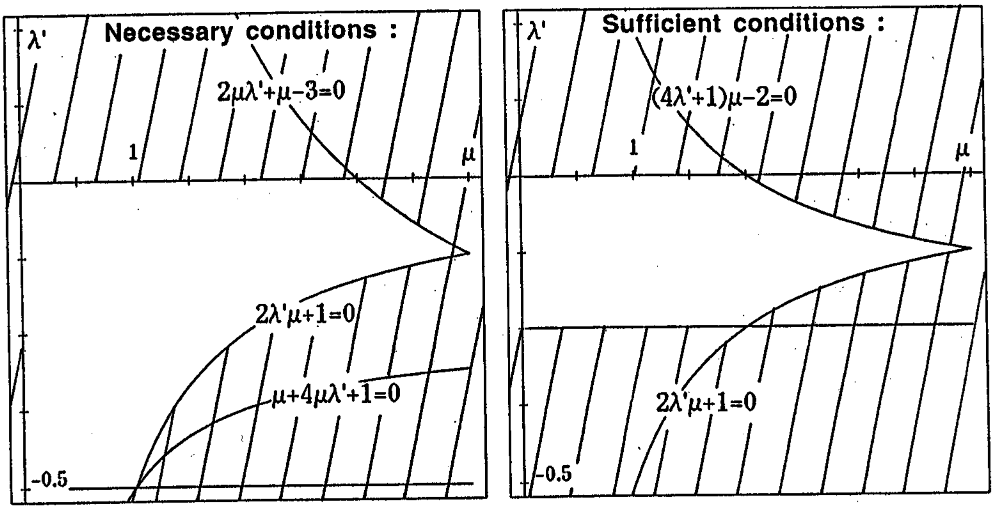
$$D_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ -ax+(1-b)y \end{pmatrix} \quad \text{and} \quad D_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ ax+(1-b)y \end{pmatrix}.$$

So

$$\begin{aligned} \left\| D_i\begin{pmatrix} x \\ y \end{pmatrix} \right\| &= |x \pm y| + k|\pm ax+(1-b)y| \\ &\leqslant (1+|a|)|x| + \left(\frac{1}{k} + |1-b|\right)k|y| \\ &= \left(1+\frac{|a|}{|a|+\epsilon}\right)|x| + (|a|+\epsilon+|1-b|)k|y|. \end{aligned}$$

We are sure that $\| D_i\binom{x}{y}| < 2$ whenever $|a|+\epsilon+|1-b| < 2$. So that a sufficient condition to get $\| D_i \| < 2$ is:

$$|a|+|1-b| < 2 \Leftrightarrow |\mu(8\lambda'+1)| + |2-\mu| < 2.$$



$\mu \in [0,4]$, $\lambda' \in [-0.5, 0.25]$

Fig. 3

So we have:

λ' \ μ	0	2	4
0	no condition		$\mu(4\lambda'+1)-2<0$
		0	
$-\frac{1}{8}$	$\lambda' > -\frac{1}{4}$		$2\mu\lambda'+1>0$

PROPOSITION 6

Necessary conditions to get f^1 continuous with $f^1 = f'$ on $[a, b]$ are:

$$|3-\mu \pm \delta| < 4, \quad \mu > 0, \quad \lambda' \geqslant -\tfrac{1}{2}, \quad 2\mu\lambda' + 1 \geqslant 0,$$

where $\delta = [(\mu+1)^2 + 32\lambda'\mu]^{1/2}$

Sufficient conditions to get f^1 continuous with $f^1 = f'$ on $[a, b]$ are:

$$|\mu(8\lambda'+1)| + |2-\mu| < 2.$$

5. Examples

The initial conditions are $f(0) = f'(0) = f(1) - 2 = f'(1) = 0$. We have built 512 points on [0, 1], which is close to the precision of a screen. Both f and f' are drawn.

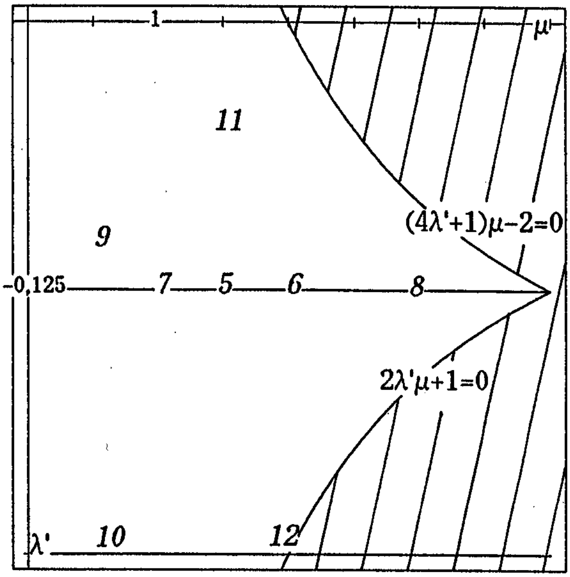

1
μ
11
9
(4λ'+1)μ−2=0
−0,125
7
5
6
8
2λ'μ+1=0
λ'
10
12

Fig. 4

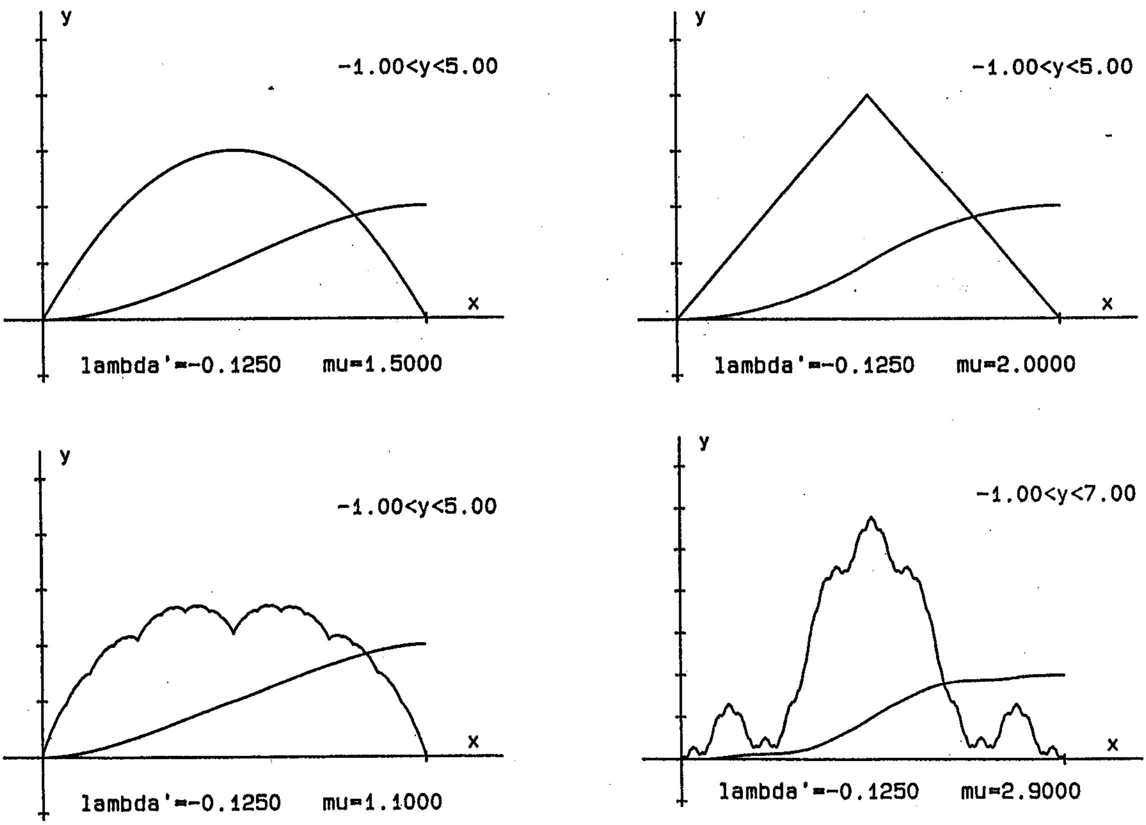

y
−1.00<y<5.00
x
lambda'=−0.1250 mu=1.5000
y
−1.00<y<5.00
x
lambda'=−0.1250 mu=2.0000
y
−1.00<y<5.00
x
lambda'=−0.1250 mu=1.1000
y
−1.00<y<7.00
x
lambda'=−0.1250 mu=2.9000

Fig. 5

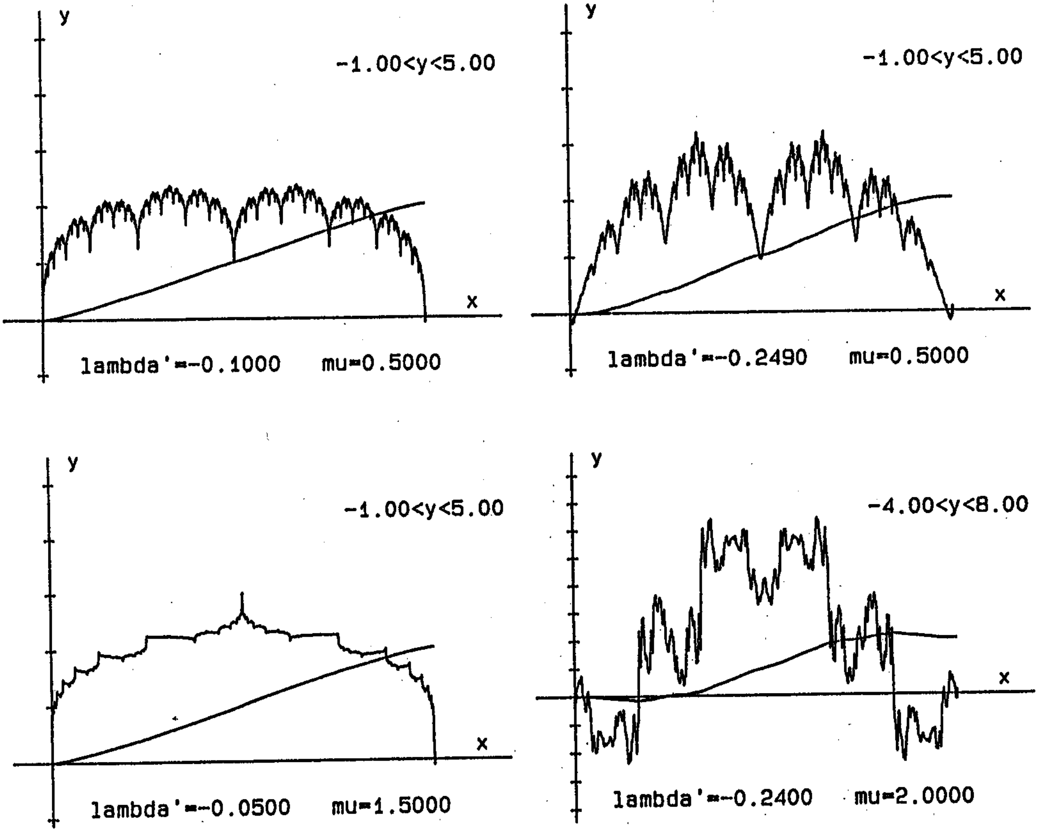


Fig. 6

The first and second examples (figs. 5, upper left and right) are, respectively, Hermite cubic and quadratic splines. The first four examples are obtained for $\lambda = -1/8 = -0.125$ and μ in $]1, 3[$ so that f^1 is Lipschitz.

Figures 6 explore the domain of sufficient conditions.

Reference

[1] Dyn, Levin and Gregory, A 4-point interpolatory subdivision scheme for curve design, CAGD 4 (1987) 257–268.